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Geometric properties of solution of a cylindrical dynamic system with impulsive state feedback control

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ABSTRACT

A kind of cylindrical dynamic system with impulsive state feedback control is formulated and investigated. Based on the qualitative properties of the corresponding continuous system, the existence of order-k ($k \in \mathbf{Z}^+$) periodic solutions of the cylindrical dynamic system with impulsive state feedback control is discussed on the cylinder with perimeter 2π. If the equilibrium of the corresponding continuous system in the rectangle coordinate system is an unstable node, then the cylindrical dynamic system has two one-side stable minimum limit sets. If the equilibrium is an unstable focus, then, for different parameters, the cylindrical dynamic system has the periodic solutions with different periods and different orders. Finally, numerical simulations are given to verify the theoretical results.

1. Introduction

In the development and management process of biological species, the implementations of some control measures depend on the state of target species, that is, only when the population or density of the target species reaches a certain threshold (e.g., economic threshold, ET), the corresponding control measures are implemented. Otherwise, no control measure is taken. This kind of control is known as state feedback control. Because those control measures have the characteristics of pulse-like actions, they are also called impulsive state feedback control. Since the impulsive state feedback control can be widely used in many biological systems and the differential equations with impulsive state feedback control can provide a natural description of the pulse-like actions, the differential equation with impulsive state feedback control receives more and more attention of researchers engaged in the study of biomathematics and other fields.

The autonomous differential equation with the term of impulsive state feedback control was called impulsive semi-dynamic system in Ref. [1] and some abstract properties (e.g., limit set) were given there. In the past years, the application of impulsive semi-dynamic system in the fields of biomathematics mainly focused on the models of pest control, microbial cultivation and disease control, the geometric properties of solution of these impulsive semi-dynamical systems were investigated clearly. For example, Refs. [2–5] studied the state-dependent impulsive systems of integrated pest management (IPM) strategies and gave the corresponding dynamic consequences. By using the method of bifurcation, Refs. [6,7] studied the impulsive state feedback control of prey–predator system and gave the existence and stability of order-1 periodic solution. Whereafter, various of prey–predator systems with impulsive state feedback control were investigated.

In the control process of microorganism culture, a turbidostat is an apparatus with feedback control system used to continuously culturing microorganisms. The dilution rate of the turbidostat can be regulated by the control system when the concentration of microorganism, detected by photoelectricity system or other devices, reaches a preset value. Based

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on the design ideas of the turbidostat, the differential equation with impulsive state feedback control was proposed in Refs. [8–10] and investigated by the existence criteria of periodic solution of a general planar impulsive autonomous system which generalized the Poincaré–Bendixson theorem [11], the conditions for the existence of order-1 periodic solution were obtained according to the preset value and the types of positive equilibrium of the corresponding continuous system. For different kinds of microorganism cultures, various differential equations with impulsive state feedback control were formulated and investigated, see Refs. [12–15].

For the disease control, Ref. [16] proposed two mathematical models with impulsive injection of insulin or its analogues for type 1 and type 2 diabetes mellitus. One model incorporated the periodic impulsive injection of insulin. The other model determined the insulin injection by closely monitoring the glucose level. The existence and stability of order-1 periodic solution were proved to explain that the perturbation by the injection in such an automated way can keep the blood glucose concentration under control.

With the application and further research of impulsive semi-dynamical system in the fields of biomathematics, some new geometric properties were found, then Ref. [17] summarized the characteristics of the biological dynamic systems with impulsive state feedback control and called it semi-continuous dynamic system. The definitions and preliminary research methods of semi-continuous dynamic system were given in Ref. [17]. Most of early researches on the biological system with impulsive state feedback control considered that the implementation of control measure only depends on one target species and the function of impulsive condition only involves one variable (see Refs. [18,19] and the papers mentioned above). But in some habitats where the resources (e.g., food, space) are limited, when the total population of the species in the habitat reaches a certain threshold, the resources will become scarce and cannot meet the need of the species to survive. At this time, some control measures, aiming at all the species in the habitat, not a single special species, should be taken to maintain the growth of all the species. To describe this kind of control condition, the function of impulsive condition will involve two or more variables.

The references mentioned above consider that the impulsive conditions are linear functions with one variable. From the geometry point of view, most of them are either the horizontal straight lines or the vertical lines in the plane. But there is few paper to discuss the case in which the function of impulsive condition is quadratic. This paper will formulate and discuss a kind of linear species system in which the function of impulsive condition is quadratic and its geometric curve is a circle, which can be considered as the reference to discuss the nonlinear biological systems in which the functions of impulsive conditions are also quadratic.

On the other hand, the differential systems given in the references mentioned above do not involve the cylindrical dynamic system. To use the existing knowledge of linear impulsive condition, this paper will formulate a kind of linear system and transform it into a cylindrical system with perimeter 2π by polar transformation, and then mainly investigate the geometric properties of solution of cylindrical system with impulsive state feedback control.

The rest of this paper is organized as follows. In Section 2, we will introduce a continuous system which can be viewed as a predator–prey system, and its semi-continuous system with impulsive state feedback control in the rectangle coordinate system. By polar transformation, the semi-continuous cylindrical system with impulsive state feedback control is formulated. The qualitative properties of the corresponding continuous system are given in Section 3. Section 4 will show that the semi-continuous cylindrical system has order-k ($k \in \mathbf{Z}^+$) periodic solutions with different periods and different orders as the parameter changes. The orbit stability of periodic solution is discussed in Section 5. Numerical simulations and discussions are given in Section 6.

2. Model formulation

Suppose that there are two species in a habitat where the food resource is limited. Denote the populations of the species by $x(t)$ and $y(t)$ at time t, respectively. For simplicity, let $x = x(t)$ and $y = y(t)$. Suppose that the species y has the negative effect on the species x and decreases the growth rate of the species x, but the species x can increase the growth rate of the species y. The relations and evolution process of two species can be described by the following system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + \delta x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x, \end{cases} \tag{2.1}$$

where $\delta \geq 0$ represents the immigration rate of species x from the outside of the habitat. System (2.1) can be viewed as a simple prey–predator system. It can be easily obtained that equilibrium $O(0, 0)$ of system (2.1) is a center for $\delta = 0$, an unstable focus for $0 < \delta < 2$ and an unstable node for $\delta \geq 2$.

From these results, we know that the population of two species x and y tend to infinite, that is, $x \to +\infty$ and $y \to +\infty$ as $t \to \infty$ for $\delta > 0$. But the food is limited, the populations of two species will decrease even tend to zero after the total population reaches a certain threshold because of the lack of food. At this time, some control measures should be taken. We suppose that the threshold satisfies $\sqrt{x^2 + y^2} = r_1,\ 0 < r_1$ and the initial state is $\sqrt{x_0^2 + y_0^2} < r_1$ where x_0 and y_0 are the initial values of x and y at the initial moment $t = t_0$. As the time t increases and the point moves along the trajectory of system (2.1), if x and y satisfy $\sqrt{x^2 + y^2} = r_1$, then some control measures can be taken and suppose that the control

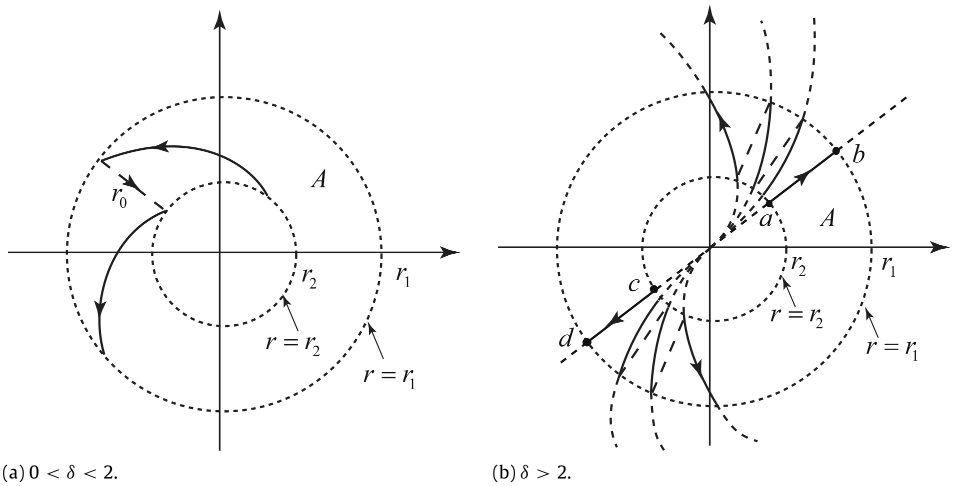


Fig. 1. Sketch map of the trajectory, the impulsive set and its image set of system (2.2).

measures make the populations of two species decrease to $\sqrt{x^2+y^2} = r_2,\ 0 < r_2 < r_1$ where r_1 and r_2 are the finite real numbers. Furthermore, we have the following system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + \delta x, & \\ & \sqrt{x^2+y^2} < r_1, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x, & \\ \Delta r = -\bar{r}, & \sqrt{x^2+y^2} = r_1 \\ r_0 = \sqrt{x_0^2+y_0^2} < r_1, & \end{cases} \tag{2.2}$$

where $\Delta r = -\bar{r} = r_2 - r_1 = \sqrt{(x^+)^2 + (y^+)^2} - \sqrt{x^2+y^2}$ and

$$x^+ = x(t_k^+) = \lim_{h\to 0^+} x(t_k + h), \qquad y^+ = y(t_k^+) = \lim_{h\to 0^+} y(t_k + h).$$

Here, $\Delta r = -\bar{r}$ is called the impulsive function which can also be written as $r^+ = r_1 - \bar{r} = r_2$. $\sqrt{x^2+y^2} = r_1$ is called the function of impulsive condition, which is of circle type [20], $M' = \{(x, y)|\sqrt{x^2+y^2} = r_1\}$ is the impulsive set and $N' = \{(x, y)|\sqrt{x^2+y^2} = r_2\}$ is the image set of the impulsive set M', they are bounded.

The trajectory, the impulsive set and its image set of system (2.2) can be seen in Fig. 1, where the circle with radius $r = r_1$ is the impulsive set and the circle with radius $r = r_2$ is the image set of the impulsive set. Fig. 1(a) gives the case of $0 < \delta < 2$ in which the equilibrium $O(0, 0)$ of system (2.1) is an unstable focus. Fig. 1(b) is the case of $\delta > 2$ in which the equilibrium $O(0, 0)$ is an unstable node. It is easily known that the annular region A between two circles ($r_2 \le \sqrt{x^2+y^2} \le r_1$) is the invariant set of system (2.2).

To use the existing knowledge of linear impulsive conditions, we transform the function of impulsive condition $\sqrt{x^2+y^2} = r_1$ into $r = r_1$ by polar transformation $x = r\cos\theta$ and $y = r\sin\theta$. After polar transformation, systems (2.1) and (2.2) become the following systems

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = \delta r\cos^2\theta, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 1 - \delta\sin\theta\cos\theta, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \left.\begin{aligned} &\dfrac{\mathrm{d}r}{\mathrm{d}t} = \delta r\cos^2\theta, \\ &\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 1 - \delta\sin\theta\cos\theta, \end{aligned}\right\} & r < r_1, \\ \left.\begin{aligned} &\Delta r = -\bar{r}, \\ &\Delta\theta = 0, \end{aligned}\right\} & r = r_1, \\ r_0 < r_1. & \end{cases} \tag{2.4}$$

For system (2.4), the impulsive function is $\Delta r = -\bar{r}$ (or $r^+ = r_1 - \bar{r} = r_2$) and $\Delta\theta = 0$, the function of impulsive condition is $r = r_1$, the impulsive set is $M = \{(\theta, r)|r = r_1\}$ and its image set is $N = \{(\theta, r)|r = r_2\}$, they are bounded.

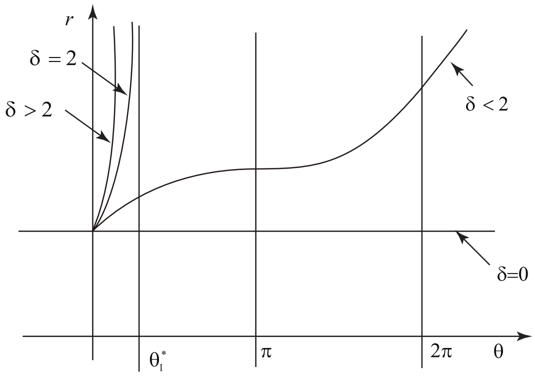


Fig. 2. Different trajectories of system (2.3) for different δ.

Generally, systems (2.3) and (2.4) can be viewed as polar equations where $r = r(t)$ is the polar radius, $\theta = \theta(t)$ is the polar angle, $r = 0$ corresponds to the equilibrium $O(0, 0)$ of system (2.1).

Since the right-hand functions of system (2.3) are periodic solutions of period 2π with respect to θ, if the lines $\theta = 0$ and $\theta = 2\pi$ are glued together, then a cylinder $S^1 \times \mathbf{R}^1$ with perimeter 2π can be formed in the rectangle coordinate system, furthermore, system (2.3) can be considered as a cylindrical dynamic system . Therefore systems (2.3) and (2.4) can be investigated on the cylinder which is a strip region in the rectangular coordinate system. The strip region parallels to the r-axis and its width is 2π.

In the following, the existence of order-k ($k \in \mathbf{Z}^+$) periodic solution of system (2.4) will be discussed in the phase plane of rectangular coordinate system. The definitions of order-k ($k \in \mathbf{Z}^+$) periodic solution can be found in Appendix A.

3. Qualitative analysis of system (2.3)

If $\delta = 0$, then system (2.3) is of center type. If we regard system (2.3) as a polar system, then the trajectories of system (2.3) are closed. If we regard it as a rectangular coordinate system, then the trajectories are the horizontal lines paralleling to the θ-axis for $\delta = 0$ (see Fig. 2). With the value of parameter δ increasing from 0, the trajectories become the curves and $r \to \infty$ as $t \to \infty$. In the following, we will discuss the properties of solution of system (2.3) in the plane of rectangular coordinate and first consider the case of $\theta \in [0, 2\pi]$ (see Fig. 2).

From the second equation of system (2.3), it follows that $\sin(2\theta) = \frac{2}{\delta}$ if $1 - \delta \sin\theta\cos\theta = 1 - \frac{1}{2}\delta\sin(2\theta) = 0$, which implies that system (2.3) has no asymptotic direction for $0 < \delta < 2$ since $\frac{2}{\delta} > 1$.

If $\delta \geq 2$, then from $\sin(2\theta) = \frac{2}{\delta}$ we can obtain that $\theta_1 = \frac{1}{2}\arcsin\left(\frac{2}{\delta}\right)$ and $\theta_2 = \frac{1}{2}\arcsin\left(\frac{2}{\delta}\right) + \pi$. In particular, if $\delta = 2$, then $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{5\pi}{4}$. Here, θ_1 and θ_2 correspond to two asymptotic directions of system (2.3).

On the other hand, it can be obtained from system (2.3) that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\delta r\cos^2\theta}{1 - \delta\sin\theta\cos\theta} = \frac{\delta r}{\left(\tan\theta - \frac{\delta}{2}\right)^2 + \left(1 - \frac{\delta^2}{4}\right)}. \tag{3.1}$$

Let $g(\theta) = (\tan\theta - \frac{\delta}{2})^2 + (1 - \frac{\delta^2}{4})$. If $\delta \geq 2$, then it follows from $g(\theta) = 0$ that

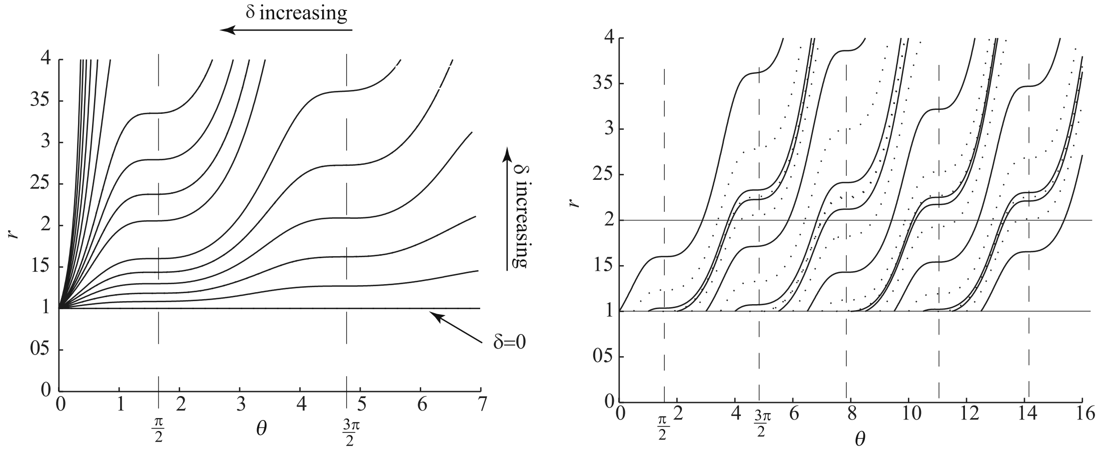
$$\theta = \arctan\left(\frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} - 1}\right) = \arctan\left(\frac{1}{2}(\delta \pm \sqrt{\delta^2 - 4})\right).$$

Therefore, there also exist two asymptotic directions $\theta^*_{1,2} = \arctan(\frac{1}{2}(\delta \pm \sqrt{\delta^2 - 4})) \in [0, 2\pi]$ along which $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ tends to infinity. It can be proved that $\theta^*_1 = \theta_1$ and $\theta^*_2 = \theta_2$ (see Appendix B).

If $\delta < 2$, since $g(\theta) \neq 0$, then $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ exists and is bounded. In particular, $g(\theta) \to \infty$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} \to 0$ for $\theta = \frac{\pi}{2}$. Therefore, we know that $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ decreases from δr to 0 for $\theta \in [0, \frac{\pi}{2}]$, increases from 0 to δr for $\theta \in [\frac{\pi}{2}, \pi]$, decreases from δr to 0 for $\theta \in [\pi, \frac{3\pi}{2}]$ and then increases from 0 to δr for $\theta \in [\frac{3\pi}{2}, 2\pi]$ (see Fig. 3). Fig. 3 gives the illustrations of the trajectories of system (2.3) with parameter δ and the initial value of θ varying.

4. Qualitative analysis of system (2.4)

In the following, we will discuss the geometric properties of solution of system (2.4) with parameter δ varying. According to the above analyses, when $\delta = 0$, all the trajectories of system (2.4) are the horizontal lines in the (θ, r)-plane of rectangular coordinate and the impulsive effect does not occur for $r < r_1$. Therefore, we will discuss the following two cases of $\delta \geq 2$ and $0 < \delta < 2$ for the initial value $r_0 < r_1$.



(a) Different values of parameter δ. (b) Different initial values of θ.

Fig. 3. Illustration of system (2.3) with the parameter δ and the initial value of θ varying.

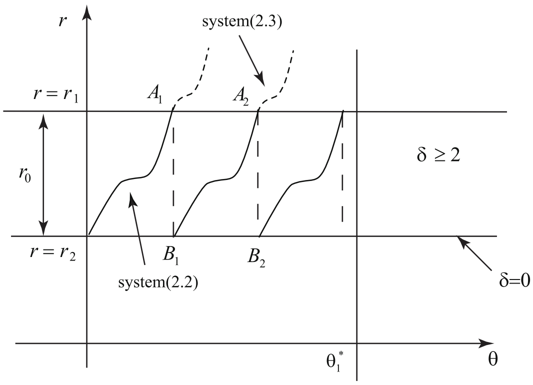


Fig. 4. Sketch map of the trajectories of system (2.4) for $\delta \geq 2$.

4.1. *Case $\delta \geq 2$*

Theorem 4.1. *System (2.4) has two one-side stable ω-limit sets for $\theta \in [0, 2\pi]$ if $\delta \geq 2$.*

Proof. If $\delta \geq 2$, then the equilibrium $O(0, 0)$ of system (2.1) (corresponding to the point $r = 0$ of system (2.3)) is an unstable node. Regard (θ, r) as the point of rectangular coordinate, see Fig. 4. Without loss of generality, let the initial point be $(0, r_2)$. Denote the trajectory starting from $(0, r_2)$ by l. Since $\frac{dr}{dt} > 0$, $\frac{d\theta}{dt} > 0$ and $\delta \geq 2$, as the time t increases from the initial moment t_0 and the point moves from $(0, r_2)$, there exists a moment t_1 such that the point along the trajectory l reaches $A_1(\theta_1, r_1)$ where $\theta_1 = \theta(t_1)$, then jumps to $B_1(\theta_1^+, r_2)$ under the impulsive effect where $\theta_1^+ = \theta(t_1^+)$ (see Fig. 4). Sequently, it continues to move starting from $B_1(\theta_1^+, r_2)$ and reaches $A_2(\theta_2, r_1)$ at the moment $t = t_2$, $\theta_2 = \theta(t_2)$, then jumps from A_2 to $B_2(\theta_2^+, r_2)$ under the impulsive effect where $\theta_2^+ = \theta(t_2^+)$. Repeating the process all the time as the time t increases, the sequences $\{A_k(\theta_k, r_1)\} \in M$ and $\{B_k(\theta_k^+, r_2)\} \in N$, $k = 1, 2, \ldots$ can be obtained, where $\theta_k = \theta(t_k)$, $\theta_k^+ = \theta(t_k^+)$, the corresponding moments are $\{t_k\}$ and $\{t_k^+\}$, $k = 1, 2, \ldots$, respectively [1].

Since $\frac{dr}{d\theta} = +\infty$ for $\theta = \theta_1^*$ where $\theta = \theta_1^*$ is the asymptotic line, then $\theta_k \to \theta_1^*$, $k = 1, 2, \ldots$ as $t \to \infty$. Furthermore, $\Omega_1 = \{(\theta, r) | \theta = \theta_1^*, r_2 \leq r \leq r_1\}$ is an ω-limit set of system (2.4). Since the trajectory starting from the point $(\theta_1^* + \epsilon, r_2)$, ϵ is sufficiently small, repeat the above process for $\theta > \theta_1^*$, then there also exists an ω-limit set $\Omega_2 = \{(\theta, r) | \theta = \theta_1^* + \pi, r_2 \leq r \leq r_1\}$.

For arbitrary $\epsilon_1 > 0$, $\eta > 0$, let $\eta_1 = \epsilon_1$, suppose that the initial moment is t_0, (θ_0, r) is the arbitrary initial point in the region where $r < r_1, \theta_0 < \theta_1^*$, $\theta_0 \notin \overline{B}_\eta(\theta(t_k)) \bigcup \overline{B}_\eta(\theta(t_k^+))$, $|t_0 - t_k| > \eta$, $|\theta - \theta_1^*| < \eta_1$, $B_\eta(\theta(t_k))$ is the η-neighborhood of $\theta(t_k)$, $k = 1, 2 \cdots$, where t_k is the impulsive moment. From $\frac{dr}{dt} > 0$ and $\frac{d\theta}{dt} > 0$, we know that the trajectory starting from the point (θ_0, r) tends to the line $\theta = \theta_1^*$ from the left side and does not intersect with the line $\theta = \theta_1^*$, which implies that $\rho((\theta, r), \Omega_1) = \inf |\theta - \theta_1^*| < \epsilon_1$ for $t > t_0$ where $\rho((\theta, r), \Omega_1)$ is the distance from the point (θ, r) to the limit set Ω_1. Similarly, the stability of the limit set Ω_2 can be discussed. Therefore, according to the definition of orbit stability (see Appendix C), we know that system (2.4) has two one-sided stable ω-limit sets for $\theta \in [0, 2\pi]$. This completes the proof. □

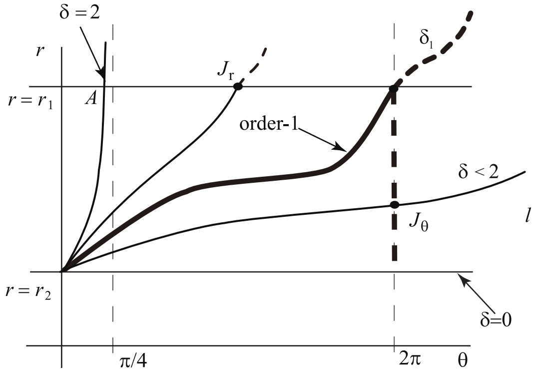


Fig. 5. Sketch map of trajectories of system (2.4) for $0 < \delta < 2$.

Let us consider Fig. 1 once again, obviously, the annular region A in Fig. 1 is the invariant set of system (2.4) (or (2.2)) but not the minimum limit set. In Fig. 1(b), the minimum limit set is the segments $\overline{ab}$ and $\overline{cd}$ which lie on the asymptotic lines ($\theta = \theta_1^*$ and $\theta = \theta_2^*$) and between two circles ($r = r_1$ and $r = r_2$).

4.2. Case $0 < \delta < 2$

Theorem 4.2. *There exists a $\delta_1 (0 < \delta_1 < 2)$ such that system (2.4) has an unique order-1 periodic solution of period 2π with respect to θ for $\delta = \delta_1$.*

Proof. From system (2.3), it follows that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\delta r \cos^2\theta}{1 - \delta \sin\theta\cos\theta} = \frac{r\cos^2\theta}{\frac{1}{\delta} - \sin\theta\cos\theta}. \tag{4.1}$$

From Eq. (4.1), we know that $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ is a strictly monotone increasing function with respect to parameter δ and has no asymptotic direction for $0 < \delta < 2$.

Since system (2.3) has an asymptotic direction θ_1^* ($\theta_1^* = \frac{\pi}{4}$) for $\delta = 2$. Without loss of generality, denote the first intersection point of the trajectory l starting from the point $(0, r_2)$ and the line $r = r_1$ by A for $\delta = 2$ (see Fig. 5). It is easy to know that $\theta_A < \frac{\pi}{4}$ for $\delta = 2$ where θ_A is the value of θ at the point A.

Denote the first intersection point of the trajectory l and the line $r = r_1$ by J_r for $0 < \delta < 2$. According to the continuous dependence of solution on parameters, if $\theta_{J_r} > \theta_A$, that is, the point J_r lies on the right side of the line $\theta = \frac{\pi}{4}$, then we know that the corresponding parameter δ is less than 2.

From the above discussions, we know that the trajectories are horizontal for $\delta = 0$. Denote the first intersection point of the trajectory l and the line $\theta = 2\pi$ by J_θ for $0 < \delta < 2$. As parameter δ is increased gradually, according to the continuous dependence of solution on parameters, it follows that J_θ moves from down to up, then there must exist a δ_1 such that the trajectory l intersects with the line $\theta = 2\pi$ at the point $(2\pi, r_1)$, which implies that system (2.4) has an order-1 periodic solution for $\delta = \delta_1$ and the period with respect to θ is 2π. This completes the proof. □

Theorem 4.3. *For $0 < \delta < 2$, system (2.4) has an order-k ($k \geq 2,\ k \in \mathbf{Z}^+$) periodic solution of period 2π with respect to θ for $\theta \in [0, 2\pi]$ where order k is finite.*

Proof. From Theorem 4.2, we know that system (2.4) has an order-1 periodic solution for $\delta = \delta_1$ and the period with respect to θ is 2π. The intersection point J_r coincides with the point $(2\pi, r_1)$ (see Fig. 5). As the value of δ increases, the point J_r moves from right to left along the line $r = r_1$. Without loss of generality, suppose that for a δ ($\delta > \delta_1$), at a certain moment, the point moving along the trajectory l starting from $(0, r_2)$ hits the impulsive set $r = r_1$ at the point $J_r(\theta_1, r_1)$, $\theta_1 < 2\pi$. Obviously, J_r lies on the left of the point $(2\pi, r_1)$ since $\delta > \delta_1$. Under the impulsive effect, the point jumps to (θ_1^+, r_2) and then moves from the point (θ_1^+, r_2). Let J'_θ be the intersection point of the trajectory l and the line $\theta = 2\pi$, if J'_θ coincides with the point $(2\pi, r_1)$, then system (2.4) has an order-2 periodic solution. If the point J'_θ lies down the point $(2\pi, r_1)$, then the value of δ can be increased continuously and the point J'_θ moves from down to up. Further, there must exist a δ ($\delta < 2$) such that (2.4) has an order-2 periodic solution since $\theta = \frac{\pi}{4}$ is the asymptotic line for $\delta = 2$. Therefore, we can denote the value of the parameter δ by δ_2 for which system (2.4) has an order-2 periodic solution (see Fig. 6).

If $\delta_2 < 2$, then the similar discussions can be continued and system (2.4) has an order-k periodic solution for the corresponding parameter $\delta = \delta_k$ ($\delta_k < 2,\ k \geq 2$).

Suppose that system (2.4) has an order-k ($k \geq 2$) periodic solution $\widetilde{l}$. The increment of θ is $\Delta\theta_1$ when $\widetilde{l}$ firstly reaches the line $r = r_1$ from the line $r = r_2$, and then the increments of θ in every impulsive interval are $\Delta\theta_2, \ldots, \Delta\theta_k$, respectively. The trajectory starting from the image point (on the line $r = r_2$) of $(k-1)$th impulsive point reaches the point $(2\pi, r_2)$

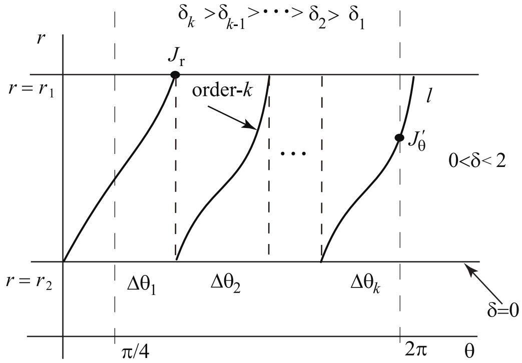


Fig. 6. The existence of order-k ($k \geq 2$) periodic solution of period 2π with respect to θ in $[0, 2\pi]$.

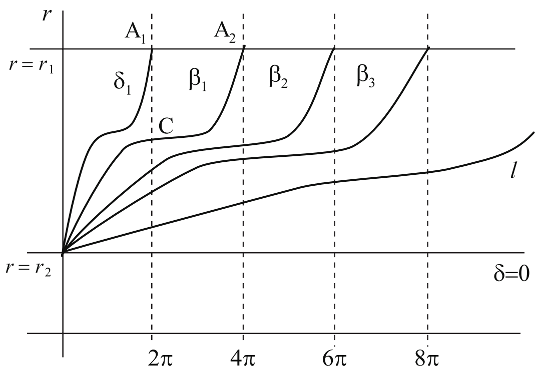


Fig. 7. The existence of order-1 periodic solution of period $2u\pi$ with respect to θ in $[0, 2u\pi]$, $u \in \mathbf{Z}^+$.

after θ undergoes the increment $\Delta\theta_k$. If the order k is infinite, that is, $k = \infty$, then there at least exist an interval such that $\Delta\theta_i \to 0$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} \to \infty$. But system (2.4) has no such asymptotic direction for $0 < \delta < 2$ and the values of $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ at every point are finite. Therefore, the order k is finite for $\theta \in [0, 2\pi]$. This completes the proof. □

4.3. Order-k periodic solution of longer period with respect to θ

From Theorems 4.2 and 4.3, we know that system (2.4) has order-k periodic solution and the number of order k is finite and $k \geq 1$ for $\theta \in [0, 2\pi]$. Suppose that the order-i periodic solutions correspond to the parameters δ_i, $i = 1, 2, \dots, k$ for $\theta \in [0, 2\pi]$, respectively, and $0 < \delta_1 < \delta_2 < \cdots < \delta_k < 2$.

For $\delta \in (0, \delta_1) \cup \cdots \cup (\delta_{k-1}, \delta_k) \cup (\delta_k, 2)$ or $\delta \in (0, 2) - \{\delta_1, \delta_2, \dots, \delta_k\}$, the existence of periodic solution need to consider in longer intervals such as $[0, 4\pi]$, $[0, 6\pi]$, ... There are two cases:

Case 1. $\delta \in (0, \delta_1)$

According to the above assumptions and Theorem 4.1, there exists a δ_1 such that system (2.4) has an order-1 periodic solution of period 2π with respect to θ for $\delta = \delta_1$ and $\theta \in [0, 2\pi]$. When $\delta < \delta_1$, the trajectory l intersects with the line $\theta = 2\pi$ at the point $C(2\pi, r)$, $r_2 < r < r_1$ (see Fig. 6), it is clear that system (2.4) has no periodic solution for $\theta \in [0, 2\pi]$. From the continuous dependence of solution on parameters, we know that the intersection point $C(2\pi, r)$ moves from up to down as δ decreases, then there must exist a $\beta_1(\beta_1 < \delta_1)$ such that the trajectory l intersects with the line $\theta = 4\pi$ at the point $A_2(4\pi, r_1)$ for $\delta = \beta_1$. Furthermore, there is an order-1 periodic solution of period 4π with respect to θ for $\delta = \beta_1 < \delta_1$ (see Fig. 7).

Similarly, if $\delta < \beta_1$, then there exists a $\beta_2(\beta_2 < \beta_1)$ such that system (2.4) has an order-1 periodic solution of period 6π with respect to θ for $\delta = \beta_2$. The same discussions can be continued and system (2.4) has order-1 periodic solutions of period $2u\pi$, $u \in \mathbf{Z}^+$ with respect to θ.

Case 2. $\delta \in (\delta_1, \delta_2) \cup \cdots \cup (\delta_{k-1}, \delta_k) \cup (\delta_k, 2)$

For $\delta \in (\delta_i, \delta_{i+1})$, $i = 1, 2, \dots, k-1$ or $\delta \in (\delta_k, 2)$, the trajectories are similar to that given in Fig. 8. The trajectory l intersects with the line $\theta = 2\pi$ at the point $A_1(2\pi, r_{A_1})$, $r_2 < r_{A_1} < r_1$ after i times impulsive effects. As θ increases, the trajectory l intersects with the line $\theta = 4\pi$ at the point $A_2(4\pi, r_{A_2})$. It is clear that there are i and $\delta_{i1}(\delta_{i1} \in (\delta_i, \delta_{i+1}))$ such that $r_{A_2}|_{\theta=2\pi} = r_1$, then for this δ_{i1}, there exists a periodic solution of period 4π with respect to θ, and its order is $2i + 1$. If $r_{A_2} \neq r_1$, then $r_{A_2} > r_{A_1}$, and the similar discussion can be continued, there must exist a δ_{iu} and the interval $\theta \in [0, 2u\pi]$, $u \in \mathbf{Z}^+$ such that $r_{A_u} = r_1$, which means that there is a periodic solution. The order is $2ui + 1$ and the period with respect to θ is $2u\pi$.

Furthermore, for $\delta \in (0, \delta_1) \cup \cdots \cup (\delta_{k-1}, \delta_k) \cup (\delta_k, 2)$, there exist $\delta_{11}, \delta_{21}, \ldots, \delta_{k1}$ such that system (2.4) has an order-$(2i+1)$ periodic solution in every corresponding parameter interval.

4.4. Maximum period and maximum order for $\theta \in \mathbf{R}^+$

Suppose that system (2.4) has a periodic solution with maximum order and maximum period with respect to θ for $\theta \in \mathbf{R}^+$ and $0 < \delta < 2$. Firstly, suppose that the maximum order is $\overline{k}$, $\overline{k} \in \mathbf{Z}^+$, the corresponding parameter is $\delta = \overline{\delta}_{\overline{k}} < 2$ and the period with respect to θ is $2\overline{u}\pi$, $\overline{u} \in \mathbf{Z}^+$. Since $\delta = \overline{\delta}_{\overline{k}} < 2$, then for $\delta \in (\overline{\delta}_{\overline{k}}, 2)$, system (2.4) has no periodic solution in the interval $\theta \in [0, 2\overline{u}\pi]$, $\overline{u} \in \mathbf{Z}^+$. According to the above discussions, we know that there is a δ ($\delta \in (\overline{\delta}_{\overline{k}}, 2)$) such that system (2.4) has order-$(2\overline{k}+1)$, $\overline{k} \in \mathbf{Z}^+$ periodic solution in the longer interval (e.g., $\theta \in [0, 2(\overline{u}+1)\pi]$, $\overline{u} \in \mathbf{Z}^+$), which is a contradiction.

Secondly, suppose that system (2.4) has an order-1 periodic solution with maximum period $2\overline{u}_1\pi$, $\overline{u}_1 \in \mathbf{Z}^+$ with respect to θ and the corresponding parameter $\delta = \overline{\delta}_1$. But for $\delta \in (0, \overline{\delta}_1)$, system (2.4) has no order-1 periodic solution in $\theta \in [0, 2\overline{u}_1\pi]$, $\overline{u}_1 \in \mathbf{Z}^+$. Similarly, the order-1 periodic solution will exist in the longer interval, which also concludes a contradiction. Therefore, we have the following proposition.

Proposition 4.1. *System (2.4) has no periodic solution with maximum period and maximum order for $\theta \in \mathbf{R}^+$ and $0 < \delta < 2$.*

Remark 4.1. Let us consider system (2.4) in the polar coordinate system (see Fig. 1). Since every trajectory of system (2.4) starting the points in the region where $r < r_1$ will hit the circle $r = r_1$, it is sufficient to consider only the trajectories starting from the point on the circle $r = r_2$. For arbitrary $\theta = \theta_0$, the trajectory starting from the point (θ_0, r_2) reaches the point (θ_1, r_1) where $\theta_1 = \theta_0 + \Delta_1$ and the impulsive effect switches it to (θ_1^+, r_1). Then the motion continues along the trajectory until it hits the circle $r = r_1$ again at the point (θ_2, r_1) where $\theta_2 = \theta_1 + \Delta_2 = \theta_0 + \Delta_1 + \Delta_2$. It is easy to see that after the kth encounter, the existence of order-k ($k \in \mathbf{Z}^+$) period solution is to consider the point (θ_k, r_1) where $\theta_k = \theta_0 + \Delta$ and $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_k$. Therefore, in order to study the motion, it is sufficient to consider the distribution of all points $\theta_k = \theta_0 + \Delta$ (see Ref. [20]).

If $\frac{\Delta}{2\pi}$ is a rational number, that is, $\Delta = 2\pi(p/q)$ where p and q are positive integers relatively prime to each other, then $\theta_q = \theta_0 + \Delta_1 + \cdots + \Delta_q = \theta_0 (\mathrm{mod}\ 2\pi)$. Thus the trajectory is an order-q periodic solution (see Ref. [20]).

Suppose that $\frac{\Delta}{2\pi}$ is not a rational number. Dividing the circle $r = r_1$ into n equal arcs and the length of each arc is $\frac{2\pi}{n}$. Since $\frac{\Delta}{2\pi}$ is not rational, then it never happens that $\theta_k = \theta_0 (\mathrm{mod}\ 2\pi)$. Among the first $(n+1)$th points $\theta_0, \theta_1, \ldots, \theta_n$, where $\theta_n = \theta_0 + \Delta_1 + \cdots + \Delta_n$, there are at least two points on the same arc, say (θ_m, r_1) and (θ_l, r_1) where $\theta_m = \theta_0 + \Delta_1 + \cdots + \Delta_m$ and $\theta_l = \theta_0 + \Delta_1 + \cdots + \Delta_l$, $m > l$ and let $s = m - l$. The increments $\Delta_{ml} = (\Delta_{l+1} + \cdots + \Delta_m)(\mathrm{mod}\ 2\pi)$ is less than $\frac{2\pi}{n}$. Let $\epsilon > 0$ be arbitrary and for n sufficiently large, we can obtain that $\frac{2\pi}{n} < \epsilon$. Furthermore, in any ϵ-neighborhood of any point on the circle $r = r_1$, there is at least one element of the set $\{\theta_0 + \Delta_{ml}\}$, which implies that the trajectories starting from the point in the region where $r < r_1$ are everywhere dense in the annulus where $r_2 < r < r_1$ (i.e. the trajectories fill up the annulus densely).

5. Orbit stability of periodic solution

Lemma 5.1 (*Analogue of Poincaré Criterion [20,21]*). *The T-periodic solution $x = \xi(t)$, $y = \eta(t)$ of the system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), & \\ & \phi(x,y) \neq 0, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), & \\ \Delta x = a(x,y), & \\ \Delta y = b(x,y), & \phi(x,y) = 0 \end{cases} \tag{5.1}$$

is orbitally asymptotically stable and enjoys the property of asymptotic phase if the multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left[\int_0^T \frac{\partial P}{\partial x}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial y}(\xi(t),\eta(t))\mathrm{d}t\right],$$

$$\Delta_k = \frac{P_+\left(\frac{\partial b}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\right) + Q_+\left(\frac{\partial a}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}}.$$

P, Q, $\frac{\partial a}{\partial x}$, $\frac{\partial a}{\partial y}$, $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$, $\frac{\partial a}{\partial x}$, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ are calculated at the point $(\xi(\tau_k), \eta(\tau_k))$, and $P_+ = P(\xi(\tau_k^+), \eta(\tau_k^+))$, $Q_+ = Q(\xi(\tau_k^+), \eta(\tau_k^+))$.

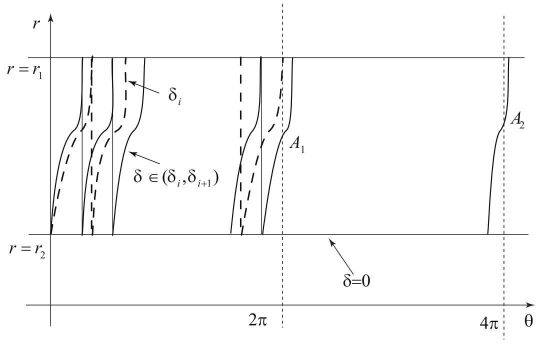


Fig. 8. The existence of order-k periodic solution of system (2.4) in the interval $[0, 4\pi]$.

Suppose that system (2.4) has an order-kT-periodic solution $(\widetilde{\theta}(t), \widetilde{r}(t))$ of period $2u\pi$ with respect to θ, $k, u \in \mathbf{Z}^+$, its period with respect to the time is T. Denote the initial moment by τ_0, the impulsive moments by $\tau_1, \tau_2, \ldots, \tau_k$, at which the trajectory l hits the line $r = r_1$. It easily follows that $T = \tau_k$ and $\theta(\tau_k) = \theta(\tau_0 + T) = 2u\pi$. From Lemma 5.1 and Ref. [20], we can obtain that

$$\frac{\partial b}{\partial \theta} = 0, \qquad \frac{\partial \phi}{\partial r} = 1, \qquad \frac{\partial b}{\partial r} = 0, \qquad \frac{\partial \phi}{\partial \theta} = 0, \qquad \frac{\partial a}{\partial r} = 0, \qquad \frac{\partial a}{\partial \theta} = 0,$$

$$\frac{\partial P}{\partial r}(\widetilde{\theta}(t), \widetilde{r}(t)) = \delta \cos^2(\widetilde{\theta}), \qquad \frac{\partial Q}{\partial \theta}(\widetilde{\theta}(t), \widetilde{r}(t)) = -\delta(2\cos^2(\widetilde{\theta}) - 1),$$

$$\Delta_k = \frac{P^+}{P} = \frac{\delta \widetilde{r}^+ \cos^2(\widetilde{\theta}^+)}{\delta \widetilde{r} \cos^2(\widetilde{\theta})} = \frac{\widetilde{r}^+}{\widetilde{r}} = \frac{r_2}{r_1},$$

$$\exp\left[\int_{\tau_i}^{\tau_{i+1}} \frac{\partial P}{\partial r}(\widetilde{\theta}(t), \widetilde{r}(t)) + \frac{\partial Q}{\partial \theta}(\widetilde{\theta}(t), \widetilde{r}(t)) \mathrm{d}t\right] = \exp\left[\int_{\tau_i}^{\tau_{i+1}} \delta(1 - \cos^2(\widetilde{\theta}))\mathrm{d}t\right], \quad i = 0, 1, 2, \ldots, k-1.$$

From the first equation of system (2.4), we have that $\frac{\mathrm{d}r}{r} = \delta \cos^2(\theta)\mathrm{d}t$. Thus, it follows that

$$\mu_2 = \left(\frac{r_2}{r_1}\right)^k \exp\left[\int_0^T \delta(1 - \cos^2(\widetilde{\theta})(t))\mathrm{d}t\right] = \left(\frac{r_2}{r_1}\right)^{k+1} \exp(\delta T).$$

From Lemma 5.1, it is easily to know that $|\mu_2| < 1$ if $\left(\frac{r_2}{r_1}\right)^{k+1} \exp(\delta T) < 1$. Therefore, we have the following proposition.

Proposition 5.1. *If system (2.4) has an order-kT-periodic solution $(\widetilde{\theta}(t), \widetilde{r}(t))$ of period $2u\pi$ with respect to θ, $k, u \in \mathbf{Z}^+$, then the periodic solution is orbitally asymptotically stable and enjoys the property of asymptotic phase if*

$$\left(\frac{r_2}{r_1}\right)^{k+1} \exp(\delta T) < 1.$$

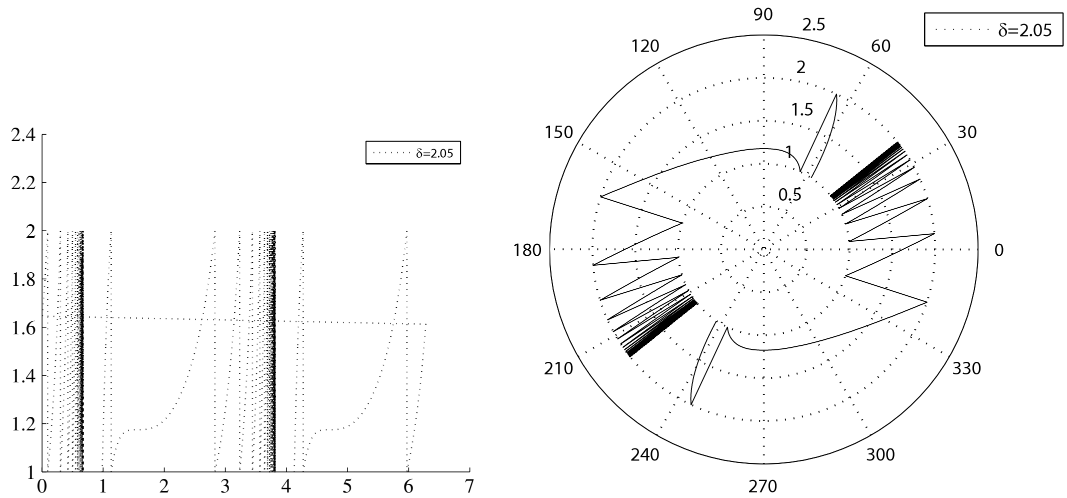
6. Numerical simulations and conclusions

From the discussions in Section 4, we know that the solutions of system (2.4) (corresponding to system (2.2)) have different geometric properties for different parameters δ. System (2.4) has two one-side stable ω-limit sets for $\theta \in [0, 2\pi]$ if $\delta \geq 2$. For $0 < \delta < 2$, we know from Remark 4.1 that system (2.4) has order-q periodic solution if the increment $\frac{\Delta}{2\pi}$ is a rational number, that is, $\Delta = 2\pi(p/q)$ where p and q are positive integers relatively prime to each other. If $\frac{\Delta}{2\pi}$ is not a rational number, then the trajectories starting from the point in the region where $r < r_1$ are everywhere dense in the annulus where $r_2 < r < r_1$.

In order to verify those theoretical results, we let $r_1 = 2$, $r_2 = 1$. For $\delta > 2$, the numerical simulations can be seen in Fig. 9 where (a) and (b) are the phase portraits of cylindrical coordinates and the polar coordinates, respectively. From Fig. 9, we can easily find that the trajectories tend to two limit sets, respectively, and the limit sets are one-side stable.

Fig. 10 gives the numerical results of periodic solutions of system (2.4) with order 1–9 and period 2π with respect to θ for $0 < \delta < 2$. The corresponding values of parameter δ are 0.2185(order 1), 0.409(order 2), 0.628(order 3), 0.807(order 4), 0.966(order 5), 1.118(order 6), 1.226(order 7), 1.332(order 8), 1.410(order 9), respectively. From Fig. 10, we can see that if the periodic solution exists, the number of the order is increasing with δ increasing for $0 < \delta < 2$ and $\theta \in [0, 2\pi]$. For convenience of comparison, some figures in Fig. 10 perform the rotation transformation with respect to the initial value, which does not affect to show the existence of the corresponding periodic solution.

For $\delta = 0.409$, system (2.4) has an order-2 periodic solution of period 2π with respect to θ (see Fig. 10(b)). For $\delta > 0.409$, Fig. 10(d, f, h) show that system (2.4) has order-4, 6, 8 periodic solution of period 2π with respect to θ which



(a) Cylindrical coordinate. (b) Polar coordinate.

Fig. 9. The numerical simulation of system (2.4) for $\delta > 2$.

corresponds to $\delta = 0.807, 1.118, 1.332$, respectively. For the parameter δ in the intervals between those values (e.g., $\delta = 0.628 \in (0.409, 0.807)$), system (2.4) has order-3 periodic solution of period 2π with respect to θ, but no order-1 or order-2 periodic solution of period π with respect to θ (see Fig. 10(c)). The similar results can be found in the other parameter intervals.

For $\delta = 0.2185 < 0.409$, system (2.4) has an order-1 periodic solution of period 2π with respect to θ. According to the theoretical results, there will exist an order-1 periodic solution of period 4π with respect to θ if $\delta < 0.2185$. Fig. 11 gives the numerical simulations under the cylindrical coordinate and the polar coordinate, respectively. Fig. 11(1) shows that there is an order-1 periodic solution of period 4π with respect to θ for $\delta = \delta_1 = 0.111$. With δ decreasing further, there is an order-1 periodic solution of period 6π with respect to θ for $\delta = \beta_1 = 0.0735 < 0.111$ (see Fig. 11(2)). Furthermore, when $\delta < 0.0735$, there is no periodic solution in $\theta \in [0, 6\pi]$, but system (2.4) can have order-1 periodic solution in the intervals $\theta \in [0, 2u\pi],\ u = 4, 5, \ldots$.

Figs. 10 and 11 have verified the existence of the case in which the increment $\frac{\Delta}{2\pi}$ is a rational number. Fig. 12 gives an example to show that the trajectories are everywhere dense in the annulus where $r_2 < r < r_1$. But it is difficult to prove that the increment $\frac{\Delta}{2\pi}$ is a rational number or not. Here only the numerical simulations are given .

Acknowledgments

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Appendix A

Definition A.1 (*Lakshmikantham et al. [1]*)**.** A triple $(X, \pi, \mathbf{R}^+)$ is said to be a semi-dynamical system if X is a metric space, $\mathbf{R}^+$ is the set of all non-negative reals and $\pi : X \times \mathbf{R}^+ \to X$ is a continuous function such that

(i) $\pi(x, 0) = x$ for all $x \in X$;
(ii) $\pi(\pi(x, t), x) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbf{R}^+$.

We denote sometimes a semi-dynamical system $(X, \pi, \mathbf{R}^+)$ by (X, π).

For any $x \in X$, the function $\pi_x : \mathbf{R}^+ \to X$ defined by $\pi_x(t) = \pi(x, t)$ is continuous and we call π_x the trajectory of x. The set $C^+(x) = \{\pi(x, t) | t \in \mathbf{R}^+\}$ is called the positive orbit of x. For any subset M of X, we let $M^+(x) = C^+(x) \bigcap M - x$ and $M^-(x) = G(x) \bigcap M - x$, where $G(x) = \bigcup\{G(x, t) | t \in \mathbf{R}^+\}$ and $G(x) = \{y | \pi(y, t) = x\}$ is the attainable set of x at $t \in \mathbf{R}^+$. Finally we set $M(x) = M^+(x) \bigcup M^-(x)$.

Definition A.2 (*Lakshmikantham et al. [1]*)**.** An impulsive semi-dynamical system $(X, \pi; M, I)$ consists of a semi-dynamical system (X, π) together with a nonempty closed subset M of X and a continuous function $I : M \to X$ such that the following properties hold:

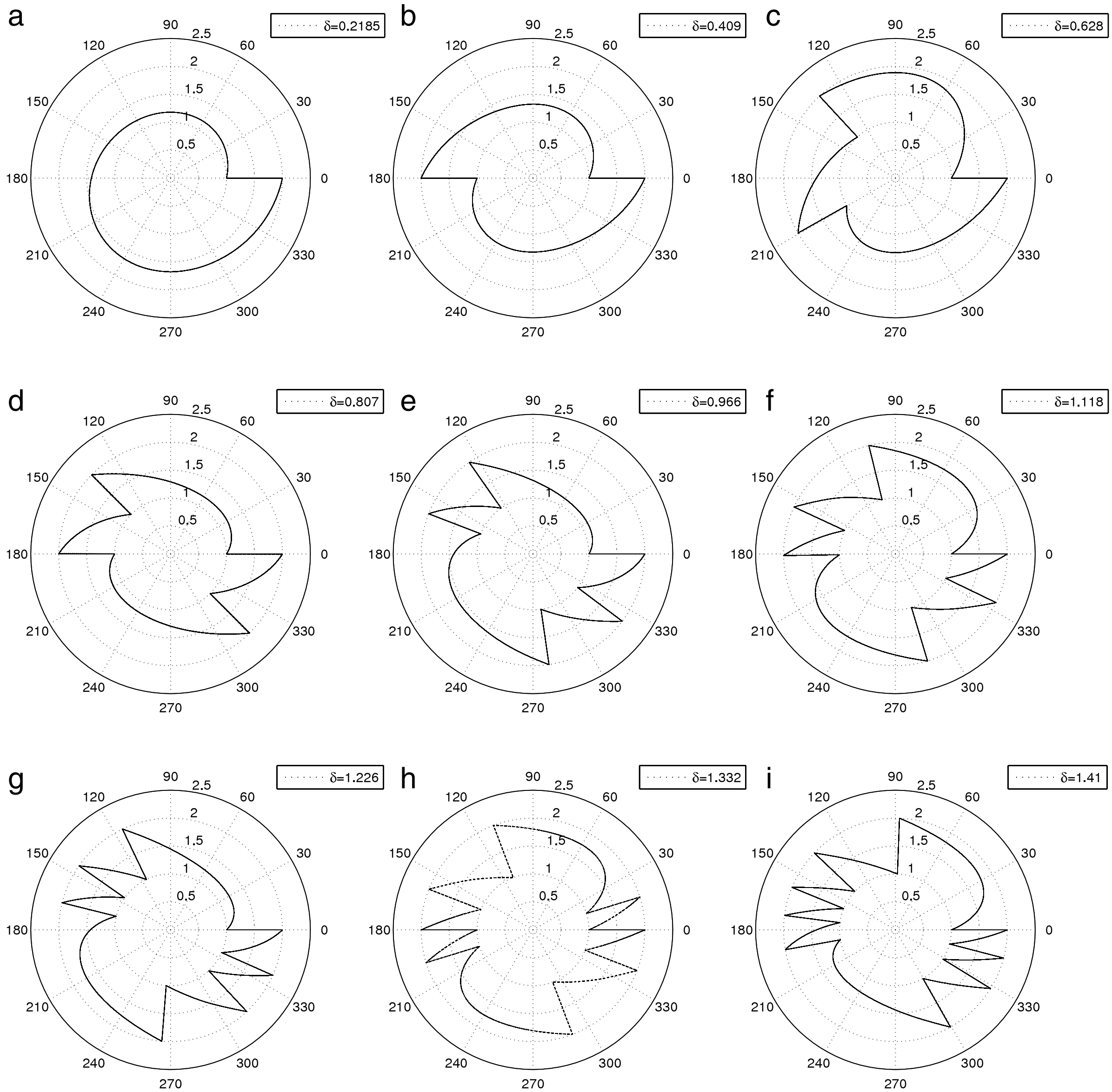


Fig. 10. Numerical simulations of order-k periodic solution of system (2.4), $k = 1, 2, \ldots, 9$.

(i) No point $x \in X$ is a limit point of $M(x)$,
(ii) $[t|G(x, t) \bigcap M \neq \varnothing]$ is a closed subset of $\mathbf{R}^+$.

According to the notations in [1], we write $N = I(M) = \{y \in X|y = I(x), x \in M$ and for any $x \in X, I(x) = x^+\}$. Here, M is called the set of impulses, I is referred to the impulsive function.

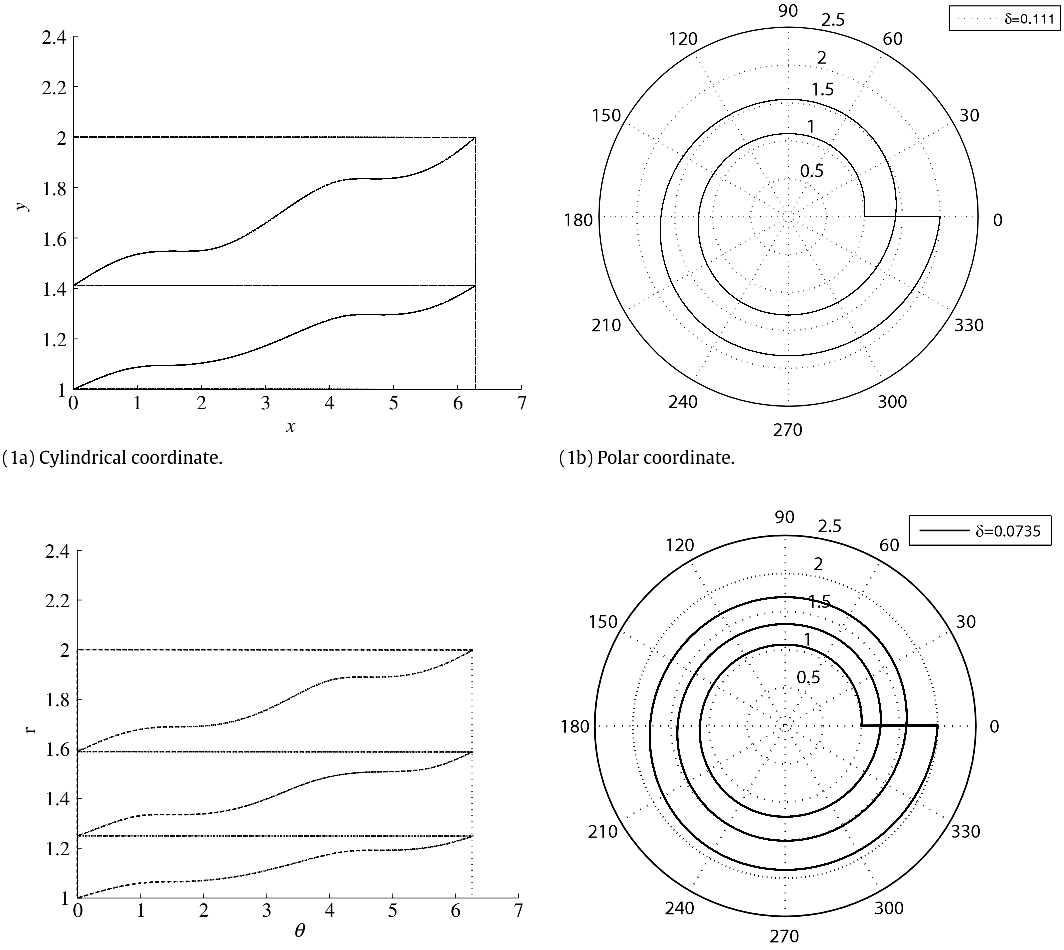
Defining a function $\Phi : X \to \mathbf{R}^+ \bigcup\{\infty\}$ as follows:

$$\Phi(x) = \begin{cases} \infty & \text{if } M^+(x) = \varnothing, \\ s & \text{if } \pi(x, t) \notin M \text{ for } 0 < t < s \text{ and } \pi(x, s) \in M. \end{cases}$$

Here s is called the time without impulse of x, i.e. s is the first time when $\pi(x, 0)$ hits M.

Definition A.3 (*Lakshmikantham et al. [1]*)**.** Let $(X, \pi; M, I)$ be an impulsive semi-dynamical system and let $x \in X$ and $x \notin M$. The trajectory of x is a function $\widetilde{\pi}_x$ defined on subset $[0, s)$ of $\mathbf{R}^+$ (s may be ∞) to X inductively as follows:

$$\widetilde{\pi}_x(t) = \widetilde{\pi}(x^+_{n-1}, t), \quad \tau_{n-1} \leq t < \tau_n,$$



(1a) Cylindrical coordinate.

(1b) Polar coordinate.

(2a) Cylindrical coordinate.

(2b) Polar coordinate.

Fig. 11. The existence of order-1 periodic solution with longer period.

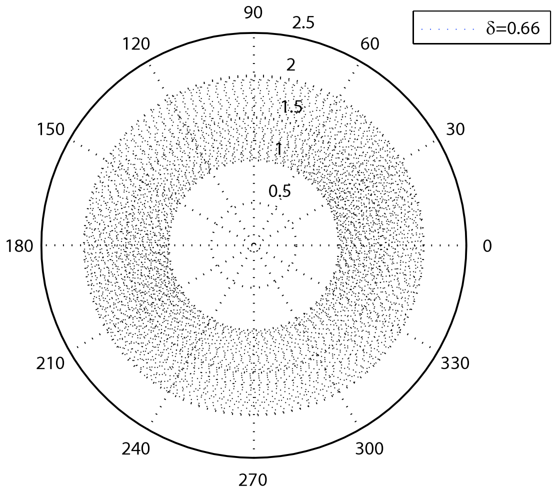


Fig. 12. The trajectories starting from the point in the region $r < r_1$ filling up the annulus $r_2 < r < r_1$ densely.

where $\{x_n\}$ is the sequence of impulse points of x, which satisfied $\pi(x_{n-1}^+, \Phi(x_{n-1}^+)) = x_n$. τ_n is the sequence of time of impulses relative to $\{x_n\}$, $\tau_n = \sum_{k=0}^{n-1} \Phi(x_k^+)$.

Definition A.4 (*Lakshmikantham et al. [1]*)**.** A trajectory $\widetilde{\pi}_x$ is said to be periodic of period τ and order k if there exist positive integers $m \geq 1$ and $k \geq 1$ such that k is the smallest integer for which $x_m^+ = x_{m+k}^+$ and $\tau = \sum_{i=m}^{m+k-1} \Phi(x_i^+)$.

Appendix B

The followings prove that $\theta_1^* = \theta_1$ and $\theta_2^* = \theta_2$ where $\theta_1 = \frac{1}{2}\arcsin\left(\frac{2}{\delta}\right)$, $\theta_2 = \frac{1}{2}\arcsin\left(\frac{2}{\delta}\right) + \pi$,

$$\theta_1{}^* = \arctan\left(\frac{1}{2}(\delta + \sqrt{\delta^2 - 4})\right)$$

and

$$\theta_2^* = \arctan\left(\frac{1}{2}(\delta - \sqrt{\delta^2 - 4})\right).$$

In fact, from the expressions of θ_1 and θ_1^*, we have that

$$\sin(2\theta_1) = \frac{2}{\delta}$$

and

$$\tan\theta_1{}^* = \frac{1}{2}(\delta + \sqrt{\delta^2 - 4}).$$

On the other hand, from

$$\begin{aligned}\sin(2\theta_1{}^*) &= 2\sin\theta_1{}^*\cos\theta_1{}^* = \frac{2\tan\theta_1{}^*}{1+\tan^2\theta_1{}^*} \\ &= \frac{\delta + \sqrt{\delta^2-4}}{1 + \frac{1}{4}(\delta^2 + \delta^2 - 4 + 2\delta\sqrt{\delta^2-4})} = \frac{2(\delta + \sqrt{\delta^2-4})}{\delta\,(\delta + \sqrt{\delta^2-4})} = \frac{2}{\delta},\end{aligned}$$

we can know that $\theta_1 = \theta_1^*$. Similarly, $\theta_2 = \theta_2^*$ can also be proved.

Appendix C

Consider the autonomous system with impulse effect [21]

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = g(x), & x \notin M, \\ \Delta x|_{x\in M} = I(x), \end{cases} \tag{C.1}$$

where $t \in \mathbf{R}$; $g, I : Q \to \mathbf{R}^n$; Ω is a domain contained in the n-dimensional Euclidean space R^n with elements $x = \mathrm{col}(x_1, \ldots, x_n)$, scalar product $(x, y) = x_1y_1 + \cdots + x_ny_n$ and norm $|x| = (x, x)^{1/2}$; M is an $(n-1)$-dimensional manifold contained in Ω.

Let $\phi(t)$ ($t \in \mathbf{R}^+ = [0, \infty)$) be a solution of system (C.1) with moments of impulse effect $\{\tau_k\} : 0 < \tau_1 < \tau_2 < \cdots$, $\lim_{t\to\infty} \tau_k = \infty$ and $L = \{x \in \mathbf{R}^n : x = \phi(t),\ t \in \mathbf{R}^+\}$. $B_\epsilon(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < \epsilon\}$ is the ϵ-neighborhood of the point $x_0 \in \mathbf{R}^n$; $J^+(t_0, x_0)$ is the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0)$ is continuable to the right; $\rho(x, L) = \inf_{y\in L} |x - y|$ is the distance from the point $x \in \mathbf{R}^n$ to the set $L \subset \mathbf{R}^n$.

Definition C.1 (*[21]*)**.** The solution $\phi(t)$ of system (C.1) is called:

1.1. orbitally stable if
$(\forall\epsilon > 0)(\forall\eta > 0)(\forall t_0 \in \mathbf{R}^+, |t_0 - \tau_k| > \eta)(\exists\delta > 0)$
$(\forall x_0 \in \Omega, \rho(x_0, L) < \delta, x_0 \notin \overline{B}_\eta(\phi(\tau_k)) \bigcup \overline{B}_\eta(\phi(\tau_k + 0)))(\forall t \in J^+(t_0, x_0))$
$\rho(x(t; t_0, x_0), L) < \epsilon$;

1.2. orbitally attractive if
$(\forall\eta > 0)(\forall t_0 \in \mathbf{R}^+, |t_0 - \tau_k| \geq \eta)(\exists\lambda > 0)$
$(\forall x_0 \in \Omega, \rho(x_0, L) < \lambda, x_0 \notin \overline{B}_\eta(\phi(\tau_k)) \bigcup \overline{B}_\eta(\phi(\tau_k + 0)))(\forall\epsilon > 0)(\exists\sigma > 0)$
$t_0 + \sigma \in J^+(t_0, x_0)(\forall t \geq t_0 + \sigma,\ t \in J^+(t_0, x_0))$
$\rho(x(t; t_0, x_0), L) < \epsilon$;

1.3. orbitally asymptotically stable if it is orbitally stable and orbitally attractive.

Definition C.2 ([21]). We shall say that the solution $\phi(t)$ of system (C.1) has the property asymptotic phase if

$$(\forall\eta > 0)(\forall t_0 \in \mathbf{R}^+, |t_0 - \tau_k| > \eta)(\exists\lambda > 0)$$
$$(\forall x_0 \in \Omega, |x_0 - \phi(t_0)| < \lambda)(\forall c \in \mathbf{R})(\forall\epsilon > 0)(\exists\sigma > |c|)$$
$$t_0 + \sigma \in J^+(t_0, x_0)(\forall t \geq t_0 + \sigma, t \in J^+(t_0, x_0), |t - \tau_k| > \eta)$$
$$|x(t + c; t_0, x_0) - \phi(t)| < \epsilon.$$

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