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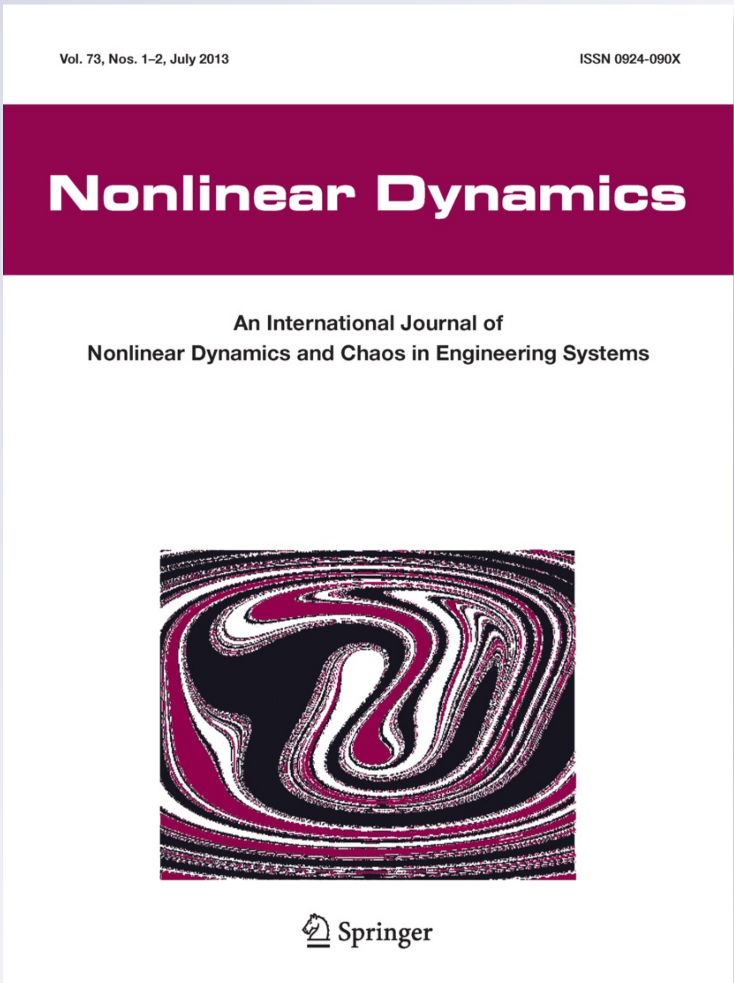

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ORIGINAL PAPER

Periodic solutions and homoclinic bifurcation of a predator–prey system with two types of harvesting

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Abstract In this paper, a predator–prey model with both constant rate harvesting and state dependent impulsive harvesting is analyzed. By using differential equation geometry theory and the method of successor functions, the existence, uniqueness and stability of the order one periodic solution have been studied. Sufficient conditions which guarantee the nonexistence of order k ($k \geq 2$) periodic solution are given. We also present that the system exhibits the phenomenon of homoclinic bifurcation under some parametric conditions. Finally, some numerical simulations and biological explanations are given.

Keywords Predator–prey system · Order k periodic solution · Successor function · Orbitally asymptotically stable · Homoclinic bifurcation

1 Introduction

Optimal management of renewable resources has become an increasingly interesting topic in the recent decades. The exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry, and wildlife management [1]. In the recent years, the harvesting effects on the dynamics of predator–prey systems have attracted lots of attentions and considerable work has been done. Different harvesting methods are applied in various situations. If the population species are harvested frequently and regularly, then it can be approximately analyzed by a constant rate harvesting [1–8]. On the other hand, if the harvesting is infrequent and periodic, we can use a periodic impulsive harvesting to model it [9–17]. For these two types of harvesting, i.e. constant rate or impulsive fashion, the population species we concerned (the predator or the prey) is harvested without knowledge of the amount of both the predator and the prey. A highly possible risk is excessive exploitation, even resource exhaustion. To improve the harvesting styles, we propose a novel idea that a reliable real time monitoring system can be introduced to estimate the number of the species. If the amount of the species satisfies specific requirements, the behavior of harvesting can be carried out, otherwise, any form of harvesting is inhibited. Such monitoring systems exist in many fields (interested readers can refer to [18–20]) and can help us to avoid excessive exploitation when we plan a long term management of a biological resource.

Brauer and Soudack [2] discussed a predator–prey system under constant rate predator harvesting and gave the region of asymptotic stability in a variety of

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scenarios. Besides, they specially studied the following system:

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{xy}{a+x}, \\ \dfrac{dy}{dt} = y\left(\dfrac{\mu x}{a+x} - d\right) - H, \end{cases} \tag{1}$$

where x and y represent the population densities of prey and predator, respectively; $H > 0$ is the constant rate at which the predators are harvested; $K > 0$ and $r > 0$, respectively, represent the carrying capacity and the intrinsic birth rate of the prey; $\mu > 0$ is the conversion rate and $d > 0$ is the death rate of the predator. The function $\frac{x}{a+x}$ denotes the predator response of Holling type II.

Brauer and Soudack [2] studied the global behavior of system (1) for some parameter values by numerical simulations. Xiao and Ruan in [1] also studied the system (1) and they mainly did a bifurcation analysis.

For predator–prey system (1), in addition to the constant rate harvesting of the predator, we can also introduce a state dependent impulsive harvesting. We assume the amount of the prey can be estimated by a monitoring system, and the monitoring data can help us decide if we harvest the predator or not. Suppose the predator which corresponds to the variable y has high commercial value, and its production is increased mainly through replenishing the quantity of its prey which corresponds to the variable x. The harvesting of the predator consists of two parts: one is a constant rate harvesting which models the frequent case (usually few in amount), and the other is an impulsive harvesting which models the infrequent case. To model this phenomenon, we can propose the following state dependent impulsive differential equations:

$$\begin{cases} \left.\begin{aligned} \dfrac{dx}{dt} &= rx\left(1 - \dfrac{x}{K}\right) - \dfrac{xy}{a+x}, \\ \dfrac{dy}{dt} &= y\left(\dfrac{\mu x}{a+x} - d\right) - H, \end{aligned}\right\} & x > h, \\ \left.\begin{aligned} x(t^+) &= x(t) + \tau, \\ y(t^+) &= (1-\beta)y(t), \end{aligned}\right\} & x = h, \end{cases} \tag{2}$$

where $h > 0$ is a threshold. When the monitoring system shows the amount of the prey is larger than h, which means the food is abundant and the predators are growing well, the development of the system coincides with the economic interest. When the amount of the prey drops to the threshold h, which means the nutrition of the predator will be deficient, we harvest the predator at rate $\beta \in (0, 1)$ and replenish a fixed amount of prey at the same time. We denote the replenishment amount as τ.

In this paper, we mainly discuss the dynamics properties of the system (2). The paper is organized as follows. In Sect. 2, some notation and definitions of the geometric theory of semi-continuous dynamical systems are provided. In Sect. 3, we mainly discuss the existence, uniqueness and orbitally stability of periodic solutions under some conditions. The paper ends with a brief discussion and some numerical simulations.

2 Preliminaries

In this section, we give some notation and definitions of the geometric theory of semi-continuous dynamical systems which will be useful for the following discussions.

Definition 1 [21] Consider the state-dependent impulsive differential equations

$$\begin{cases} \left.\begin{aligned} \dfrac{dx}{dt} &= \bar{P}(x, y), \\ \dfrac{dy}{dt} &= \bar{Q}(x, y), \end{aligned}\right\} & (x, y) \notin M\{x, y\}, \\ \left.\begin{aligned} \Delta x &= \alpha(x, y), \\ \Delta y &= \beta(x, y), \end{aligned}\right\} & (x, y) \in M\{x, y\}. \end{cases} \tag{3}$$

We define the dynamic system consisting of the solution mappings of the system (3) a semi-continuous dynamical system, denoted as (Ω, f, φ, M). We require that the initial point P of the system (3) should not be in the set $M\{x, y\}$, that is, $P \in \Omega = R_+^2 \setminus M\{x, y\}$, and the function φ is a continuous mapping that satisfies $\varphi(M) = N$. Here φ is called the impulse mapping, where $M\{x, y\}$ and $N\{x, y\}$ represent the straight lines or curves in the plane R_+^2, $M\{x, y\}$ is called the impulse set, and $N\{x, y\}$ is called the phase set.

Remark 1 For the system (2), $M = \{(x, y) \mid x = h, y \geq 0\}$, $N = \{(x, y) \mid x = h + \tau, y \geq 0\}$, and for any $(x, y) \in M$, we have $\varphi(x, y) = (h + \tau, (1 - \beta)y)$.

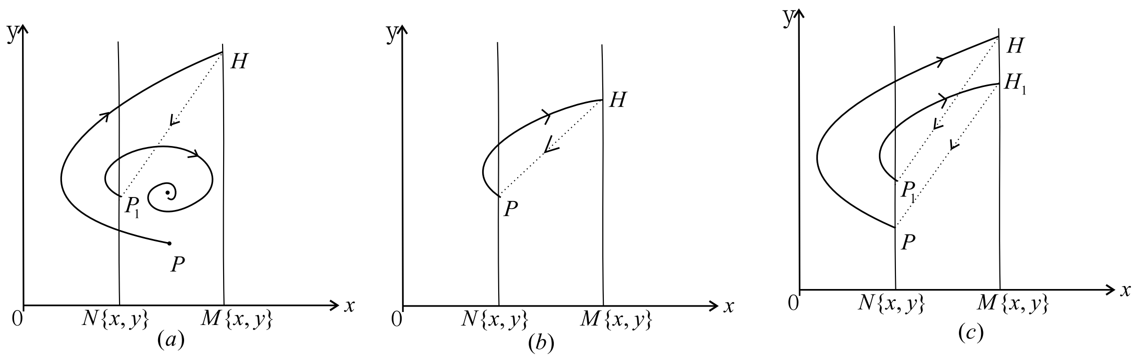

Fig. 1 (**a**) The solution mapping of the system (3). (**b**) Order one periodic solution. (**c**) Order two periodic solution

Definition 2 [21] For the semi-continuous dynamical system defined by the state-dependent impulsive differential equations (3), the solution mapping $f(P,t): \Omega \to \Omega$ consists of two parts:

(1) Let $\pi(P,t)$ denote the Poincaré mapping with the initial point P of the following system:

$$\begin{cases} \dfrac{dx}{dt} = \bar{P}(x,y), \\ \dfrac{dy}{dt} = \bar{Q}(x,y). \end{cases}$$

If $f(P,t) \cap M\{x,y\} = \emptyset$, then $f(P,t) = \pi(P,t)$.

(2) If there exists a time point T_1 such that $f(P,T_1) = H \in M\{x,y\}$, $\varphi(H) = \varphi(f(P,T_1)) = P_1 \in N\{x,y\}$ and $f(P_1,t) \cap M\{x,y\} = \emptyset$, then $f(P,t) = \pi(P,T_1) + f(P_1,t)$ (see Fig. 1(a)).

Remark 2 For (2) in Definition 2, if $f(P_1,t) \cap M\{x,y\} \neq \emptyset$, and there exists a time point T_2 such that $f(P_1,T_2) = H_1 \in M\{x,y\}$, $\varphi(H_1) = \varphi(f(P_1,T_2)) = P_2 \in N\{x,y\}$ and $f(P_2,t) \cap M\{x,y\} = \emptyset$, then $f(P,t) = \pi(P,T_1) + f(P_1,t) = \pi(P,T_1) + \pi(P_1,T_2) + f(P_2,t)$.

If $f(P_2,t) \cap M\{x,y\} \neq \emptyset, \dots, f(P_{k-1},t) \cap M\{x,y\} \neq \emptyset$ and $f(P_k,t) \cap M\{x,y\} = \emptyset$, then we can repeat the above steps and have the following form:

$$f(P,t) = \sum_{i=1}^{k} \pi(P_{i-1},T_i) + f(P_k,t), \qquad P_0 = P.$$

Definition 3 [21] If there exist a point $P \in N\{x,y\}$ and a time point T_1 such that $f(P,T_1) = H \in M\{x,y\}$ and $\varphi(H) = \varphi(f(P,T_1)) = P \in N\{x,y\}$, then $f(P,t)$ is called an order one periodic solution of the system (3) whose period is T_1 (see Fig. 1(b)). The orbit of the order one periodic solution is called an order one cycle. If there exists a singularity in the order one cycle, we call it an order one singular cycle. If the singularity is a saddle, we call it an order one homoclinic cycle.

Definition 4 [21] If there exist a point $P \in N\{x,y\}$ and a time point T_1 such that $f(P,T_1) = H \in M\{x,y\}$ and $\varphi(H) = P_1 \in N\{x,y\}$, and there also exists a time point T_2 such that $f(P_1,T_2) = H_1 \in M\{x,y\}$ and $\varphi(H_1) = P \in N\{x,y\}$, then $f(P,t)$ is called an order two periodic solution of the system (3) whose period is $T_1 + T_2$ (see Fig. 1(c)). Analogously, we can define the order k periodic solution of the system (3).

Definition 5 Suppose $\Gamma = f(P,t)$ is an order one periodic solution of the system (3). If for any $\varepsilon > 0$, there must exist $\delta > 0$ and $t_0 \geq 0$, such that for any point $P_1 \in U(P,\delta) \cap N\{x,y\}$, we have $\rho(f(P_1,t),\Gamma) < \varepsilon$ for $t > t_0$, then we call the order one periodic solution Γ is orbitally asymptotically stable.

Definition 6 [21] Suppose the impulse set M and phase set N of the system (3) are straight lines and a coordinate system can be defined in the phase set N. Let point $A \in N$ and its coordinate is a. Assume that the trajectory from the point A intersects the impulse set M at a point A', and, after impulsive effect, the point A' is mapped to the point $A_1 \in N$ with the coordinate a_1, then we call point A_1 is the order one successor point of point A, and the order one successor function of point A is $F_1(A) = a_1 - a$.

Remark 3 For system (2), we define the coordinate of point $H \in N = \{(x, y) \mid x = h + \tau, y \geq 0\}$ as its coordinate in y-axis.

Remark 4 For Definition 6, if the trajectory from the point A_1 intersects the impulse set M again at a point A_1', and, after impulsive effect, the point A_1' is mapped to the point $A_2 \in N$ with the coordinate a_2, then the point A_2 is obviously the order one successor point of point A_1, we also call point A_2 is the order two successor point of point A, and the order two successor function of point A is $F_2(A) = a_2 - a$. If the process can be repeated over and over again, then we can define the order k successor point of point A (which we denote as A_k and its coordinate is a_k) and the order k successor function of A which we denote as $F_k(A) = a_k - a$.

Lemma 1 [21] *Successor function $F_k(A)$ is continuous.*

Lemma 2 *For the systems (2), if there exist two points $A \in N$, $B \in N$ such that $F_k(A)F_k(B) < 0$, then there must exist a point $C \in N$ which is between the points A and B such that $F_k(C) = 0$, thus the system must have an order k periodic solution which passes through the point C.*

Proof By Lemma 1, we can easily see that there must exist a point $C \in N$ which is between the points A and B such that $F_k(C) = 0$. According to Definition 4, we know $\Gamma = f(C, t)$ is an order k periodic solution. That completes the proof. □

3 Existence, uniqueness and stability of periodic solutions

In this section, we mainly discuss the existence, uniqueness and stability of the order k periodic solution of the system (2) by using differential equation geometry theory and the method of successor functions. Before those discussions, we should consider the qualitative characteristics of the system (1), and we mainly discuss the conditions under which the system (1) has no periodic solution.

3.1 Qualitative analysis of the system (1)

Firstly, we consider the equilibria of system (1) in R_+^2. By setting $rx(1 - \frac{x}{K}) - \frac{xy}{a+x} = 0$ and $y(\frac{\mu x}{a+x} - d) - H = 0$, we can obtain

$$\begin{cases} y = r\left(1 - \dfrac{x}{K}\right)(a + x), \\ y = \dfrac{H(a + x)}{(\mu - d)x - ad}. \end{cases}$$

Then we have

$$f(x) = \frac{r}{K}(\mu - d)x^2 - r\left[(\mu - d) + \frac{ad}{K}\right]x + rad + H = 0.$$

Let

$$\Delta = r^2\left[(\mu - d) + \frac{ad}{K}\right]^2 - 4\frac{r}{K}(\mu - d)(rad + H),$$

then we find that if the conditions

$$\text{(H1):}\quad \mu > d, \qquad \frac{ad}{(\mu - d)} < K, \quad \text{and} \quad \left[(\mu - d) + \frac{ad}{K}\right]^2 > \frac{4(\mu - d)(rad + H)}{Kr}$$

hold, then the system (1) has two positive equilibria.

When the two positive equilibria exist, we denote them as $E_1(x_1, y_1)$, $E_2(x_2, y_2)$, where

$$x_1 = K\frac{r[(\mu - d) + \frac{ad}{K}] - \sqrt{\Delta}}{2r(\mu - d)} = \frac{K}{2} + \frac{ad}{2(\mu - d)} - \frac{K\sqrt{\Delta}}{2r(\mu - d)},$$

$$y_1 = r\left(1 - \frac{x_1}{K}\right)(a + x_1),$$

$$x_2 = K\frac{r[(\mu - d) + \frac{ad}{K}] + \sqrt{\Delta}}{2r(\mu - d)} = \frac{K}{2} + \frac{ad}{2(\mu - d)} + \frac{K\sqrt{\Delta}}{2r(\mu - d)},$$

$$y_2 = r\left(1 - \frac{x_2}{K}\right)(a + x_2).$$

Now, we begin the analysis of the stability of the equilibria of system (1).

The Jacobian matrix at $E_i, i = 1, 2$ is given by

$$J(E_i) = \begin{pmatrix} r - \frac{2rx_i}{K} - \frac{ay_i}{(a+x_i)^2} & -\frac{x_i}{a+x_i} \\ \frac{a\mu y_i}{(a+x_i)^2} & \frac{\mu x_i}{a+x_i} - d \end{pmatrix}.$$

Through calculations, we get

$$\begin{aligned} \operatorname{Det}\big(J(E_i)\big) &= -rd + \frac{2rdx_i}{K} + \frac{r\mu x_i}{a+x_i} + \frac{ady_i}{(a+x_i)^2} \\ &\quad - \frac{2\mu r x_i^2}{K(a+x_i)} \\ &= \frac{\frac{2r}{K}(\mu-d)x_i[\frac{K}{2} + \frac{ad}{2(\mu-d)} - x_i]}{(a+x_i)}, \end{aligned}$$

$$\begin{aligned} \operatorname{Tr}\big(J(E_i)\big) &= r - d - \frac{2rx_i}{K} - \frac{ay_i}{(a+x_i)^2} + \frac{\mu x_i}{a+x_i} \\ &= \frac{\frac{2r}{K}x_i[\frac{K-a}{2} + \frac{K\mu}{2r} - x_i]}{(a+x_i)} - d. \end{aligned}$$

Obviously, $\operatorname{Det}(J(E_1)) > 0$ and $\operatorname{Det}(J(E_2)) < 0$. That is to say $E_2(x_2, y_2)$ is a saddle, and $E_1(x_1, y_1)$ is an elementary and not saddle-type equilibrium.

Besides, if the condition

$$\begin{aligned} \text{(H2)}: \quad x_1 &= \frac{K}{2} + \frac{ad}{2(\mu-d)} - \frac{K\sqrt{\Delta}}{2r(\mu-d)} \\ &> \frac{K-a}{2} + \frac{K\mu}{2r} \end{aligned}$$

holds, then $\operatorname{Tr}(J(E_1)) < 0$, that is to say, $E_1(x_1, y_1)$ is a node or focus which is locally asymptotically stable.

In the following, we begin to discuss the conditions that the system (1) has no periodic solution in the interior of the first quadrant.

For the sake of simplicity, we put in dimensionless form of the system (1) by using dimensionless time $t = (a+x)\omega$, and this leads to the following system:

$$\begin{cases} \dfrac{dx}{d\omega} = x\left[ar + r\left(1 - \dfrac{a}{K}\right)x - \dfrac{r}{K}x^2 - y\right] \\ \quad = P(x, y), \\ \dfrac{dy}{d\omega} = (\mu - d)y\left[x - \dfrac{ad}{(\mu-d)} - \dfrac{H(a+x)}{(\mu-d)y}\right] \\ \quad = Q(x, y). \end{cases} \tag{4}$$

By the new approach to prove the nonexistence of limit cycle in [22], we let

$$\begin{aligned} &M(x, y) = \theta x^{-2} y^{-s}, \qquad N(x, y) = 0, \\ &B(x, y) = x^{-1} y^{-1-s}, \end{aligned}$$

where θ and s is to be determined, then we can get

$$\begin{aligned} &\mathscr{L}(M, N, P, Q) \\ &\quad = \frac{\partial(NQ)}{\partial x} - \frac{\partial(MP)}{\partial y} + \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \\ &\quad = x^{-1}y^{-1-s}\left\{-\frac{r}{K}(\theta s + 2)x^2\right. \\ &\qquad + \left[r\left(1 - \frac{a}{K}\right)(\theta s + 1) - s(\mu - d)\right]x \\ &\qquad + \theta ars + sad + \theta(1-s)y \\ &\qquad \left. + H(s+1)(a+x)y^{-1}\right\}. \end{aligned}$$

Let $s = -1$, then we have

$$\begin{aligned} &\mathscr{L}(M, N, P, Q) \\ &\quad = x^{-1}\left\{\frac{r}{K}(\theta - 2)x^2\right. \\ &\qquad + \left[r\left(1 - \frac{a}{K}\right)(1-\theta) + (\mu - d)\right]x \\ &\qquad \left. - a(\theta r + d)\right\} + 2\theta x^{-1} y. \end{aligned}$$

If there exists $\theta = \theta_0 < 0$ such that

$$\begin{aligned} g(x, \theta_0) &= \frac{r}{K}(\theta_0 - 2)x^2 \\ &\quad + \left[r\left(1 - \frac{a}{K}\right)(1 - \theta_0) + (\mu - d)\right]x \\ &\quad - a(\theta_0 r + d) < 0, \quad 0 < x < \infty, \end{aligned}$$

then we know $\mathscr{L}(M, N, P, Q) < 0$ for $(x, y) \in R_+^2$ when $M(x, y) = \theta_0 x^{-2} y$, $N(x, y) = 0$, $B(x, y) = x^{-1}$, and we get the system (4) has no periodic solution in R_+^2 by the result appeared in [22].

We now search for the conditions that guarantee there exists $\theta_0 < 0$ such that $g(x, \theta_0) < 0$, $0 < x < \infty$. We denote the discriminant of quadratic equation $g(x) = 0$ as $\Delta_1(\theta)$ which is seen as a function in θ, then

$$\begin{aligned}\Delta_1(\theta) &= \left[r\left(1-\frac{a}{K}\right)(1-\theta)+(\mu-d)\right]^2 \\ &\quad + \frac{4ra}{K}(\theta-2)(\theta r+d) \\ &= r^2\left[\left(1-\frac{a}{K}\right)^2+\frac{4a}{K}\right]\theta^2-2\left[r^2\left(1-\frac{a}{K}\right)^2\right. \\ &\quad \left.+r(\mu-d)\left(1-\frac{a}{K}\right)-\frac{2ar(d-2r)}{K}\right]\theta \\ &\quad +\left[r\left(1-\frac{a}{K}\right)+(\mu-d)\right]^2-\frac{8ard}{K}.\end{aligned}$$

Obviously, we just need to find a $\theta_0 < 0$ such that $\Delta_1(\theta_0) < 0$. It is easy to know that if the condition

$$\text{(H3):}\quad \left[r\left(1-\frac{a}{K}\right)+(\mu-d)\right]^2 < \frac{8ard}{K}$$

hold, then we have $\Delta_1(0) < 0$ and there must exist $\theta_0 < 0$ such that $\Delta_1(\theta_0) < 0$ because of the continuity of $\Delta_1(\theta)$.

Because the transformation between the system (1) and the system (4) is a homomorphism, we can get the following result according to the above discussion:

Theorem 1 *If the conditions* (H1), (H2), (H3) *are satisfied, then the system* (1) *has two equilibria: a saddle* $E_2(x_2, y_2)$ *and a locally asymptotically stable node or focus* $E_1(x_1, y_1)$, *and system* (1) *has no closed orbits in the interior of the first quadrant.*

3.2 Existence, uniqueness and stability of periodic solutions of the system (2)

According to the impulsive differential equations (2), the threshold h and the recruitment τ of the prey when people harvest the predator should satisfy the condition $0 < h < h+\tau < K$ by ecological significance. For this consideration, we have the following results.

Theorem 2 *If the conditions* (H1), (H2), (H3) *are satisfied, and* $x_1 < h < h+\tau < x_2$, *then there must exist fixed values* β^0 *and* β^* *which satisfy* $0 < \beta^0 < \beta^* < 1$ *such that for every* $\beta \in (\beta^0, \beta^*)$, *the system* (2) *has a unique order one periodic solution in region* Ω_1, *where region* Ω_1 *is the region enclosed by the* x*-axis, the impulse set* $x = h$ *and the unstable flow of the saddle* $E_2(x_2, y_2)$.

Proof According to Theorem 1, the system (1) has two equilibria: a saddle $E_2(x_2, y_2)$ and a locally asymptotically stable node or focus $E_1(x_1, y_1)$, and system (1) has no closed orbits in the interior of the first quadrant. For convenience, we denote the x-axis intersects impulse set $x = h$ and phase set $x = h+\tau$ at point A' and point B', respectively, the unstable flow of $E_2(x_2, y_2)$ intersects impulse set $x = h$ and phase set $x = h+\tau$ at point A and point O, respectively, the vertical isocline $\frac{dx}{dt} = 0$ intersects impulse set $x = h$ and phase set $x = h+\tau$ at point C and point D, respectively, and the stable flow of $E_2(x_2, y_2)$ intersects impulse set $x = h$ and phase set $x = h+\tau$ at point A_1 and point B, respectively, then the region Ω_1 is the interior of the closed curve $\overset{\frown}{E_0E_2OA} \cup \overline{ACA'} \cup \overline{A'B'E_0}$, where E_0 denotes the intersection of the unstable flow of E_2 and x-axis (see Fig. 2).

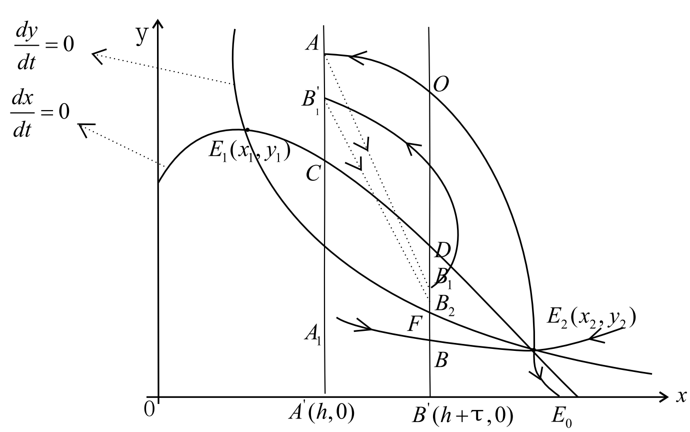

Fig. 2 The existence of order one periodic solution of the system (2)

By the impulsive conditions of the system (2), there must exist a fixed value $\beta^0 \in (0,1)$, when $\beta = \beta^0$, point A is mapped to the point D after impulsive effect, that is to say, $(1-\beta^0)y_A = y_D$; also there must exist a fixed value $\beta^* \in (\beta^0, 1)$, when $\beta = \beta^*$, point A is mapped to the point B after impulsive effect, that is to say, $(1-\beta^*)y_A = y_B$. In this paper, we denote y_H as the coordinate of point H in y-axis.

When $\beta \in (\beta^0, \beta^*)$, after impulsive effect, the point A is mapped to a point B_1 which is also the order one successor point of B, then we have $(1-\beta^*)y_A = y_B < y_{B_1} = (1-\beta)y_A < (1-\beta^0)y_A = y_D$, that is to say, point B_1 is between the point B and point D. The trajectory of the system (2) from point B_1 must intersect the impulse set $x = h$ again at a point B_1', and the point B_1' is mapped to a point B_2 after impulsive effect. Since distinct trajectories do not intersect, we can easily have $y_C < y_{B_1'} < y_A$ and $y_{B_2} = (1-\beta)y_{B_1'} < (1-\beta)y_A = y_{B_1}$ (see Fig. 2). Obviously, point B_2 is the order two successor point of point B and also the order one successor point of point B_1, then we have the following results of the order one successor function:

$$F_1(B) = y_{B_1} - y_B > 0, \qquad F_1(B_1) = y_{B_2} - y_{B_1} < 0.$$

By Lemma 2, we know that in the phase set $x = h + \tau$ there must exist a point M which is between the points B and B_1 such that $F_1(M) = 0$, then we know the system (2) has an order one periodic solution which passes through the point M.

In the following, we prove the uniqueness of the order one periodic solution. Arbitrarily choose two points A_1 and A_2 which are in the phase set $x = h + \tau$, where $y_B \le y_{A_1} < y_{A_2} \le y_D$. Then the trajectories of the system (2) from points A_1 and A_2 must intersect the impulse set $x = h$ at some points A_1' and A_2', respectively, and satisfy $y_C \le y_{A_2'} < y_{A_1'} \le y_A$. After impulsive effect, the points A_1' and A_2' must be mapped to two points in the phase set $x = h + \tau$ which we denote as A_1'' and A_2'', respectively, and $y_{A_1''} = (1-\beta)y_{A_1'}$ and $y_{A_2''} = (1-\beta)y_{A_2'}$ (see Fig. 3).

Obviously, the point A_i'' is the order one successor point of A_i, $i = 1, 2$. Then we have the order one successor functions must satisfy

$$\begin{aligned} F_1(A_2) - F_1(A_1) &= (y_{A_2''} - y_{A_2}) - (y_{A_1''} - y_{A_1}) \\ &= (y_{A_2''} - y_{A_1''}) + (y_{A_1} - y_{A_2}) < 0, \end{aligned}$$

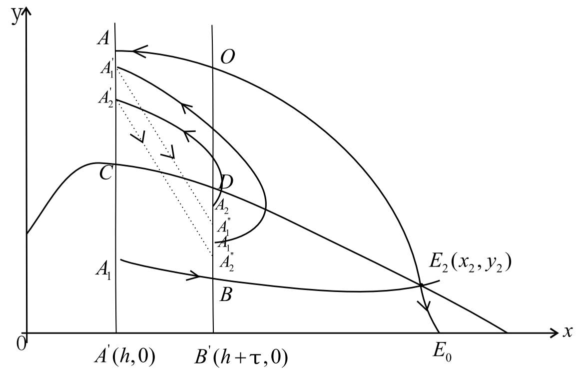

Fig. 3 The monotonicity of the successor function F_1 in the segment $\overline{BD}$

which means the order one successor function F_1 is monotonically decreasing in the segment $\overline{BD}$, thus there exists only one point $M \in \overline{BD}$ such that $F_1(M) = 0$.

For any point $H \in \overline{DO}$, the trajectory of the system (2) from point H must intersect the impulse set $x = h$ at a point H', and after impulsive effect, the point H' is mapped to a point H_1. Obviously, the point H_1 is the order one successor point of H. Since distinct trajectories do not intersect, it is easy to know $y_C < y_{H'} < y_A$ and $y_{H_1} = (1-\beta)y_{H'} < (1-\beta)y_A = y_{B_1} < y_H$, then we have $F_1(H) < 0$, which means the system (2) has no order one periodic solution passing through the point H where $H \in \overline{DO}$. Besides, for any point $H \in \overline{B'B}$, the trajectory of the system (2) from point H must ultimately pass through the x-axis and doesn't come across any impulsive effect, that is to say, the system (2) has no order one periodic solution passing through the point H where $H \in \overline{B'B}$.

To sum up, the system (2) has a unique order one periodic solution in the region Ω_1. That completes the proof. □

Theorem 3 *Under the conditions of Theorem 2, if $\beta \in (\beta^0, \beta^*)$ and $(1-\beta)r(1-\frac{h}{K})(a+h) \ge \frac{H(a+h+\tau)}{(\mu-d)(h+\tau)-ad}$, then the order one periodic solution of the system (2) is orbitally asymptotically stable, where β^0 and β^* are defined in Theorem 2.*

Proof For sake of convenience, we denote the horizontal isocline $\frac{dy}{dt} = 0$ intersects the phase set $x = h + \tau$ at point F, then $y_F = \frac{H(a+h+\tau)}{(\mu-d)(h+\tau)-ad}$. It is easy to know the isocline $\frac{dy}{dt} = 0$ is above the stable flow of $E_2(x_2, y_2)$, then we can get $y_F > y_B$

(see Fig. 2). Besides, by $y_C = r(1-\frac{h}{K})(a+h)$ and $(1-\beta)r(1-\frac{h}{K})(a+h) \geq \frac{H(a+h+\tau)}{(\mu-d)(h+\tau)-ad}$, we know $(1-\beta)y_C \geq y_F > y_B$. Obviously, for every point $H \in \overline{CA}$, we have $(1-\beta)y_H \geq (1-\beta)y_C \geq y_F > y_B$.

According to Theorem 2, the system (2) has a unique order one periodic solution that passes through the point M which is in the phase set $x = h+\tau$, and $y_B < y_M < y_{B_1}$. The trajectory of the system (2) from point B_1 must intersect the impulse set $x = h$ again at a point B_1', and after impulsive effect, the point B_1' is mapped to a point B_2 which is in the phase set $x = h+\tau$. Because distinct trajectories do not intersect, we can easily get $y_C < y_{B_1'} < y_{M'}$ and $y_B < y_{B_2} < y_M$, where M' is the impulse point of the order one periodic solution. Besides, the trajectory of the system (2) from point B_2 must intersect the impulse set $x = h$ again at a point B_2', and after impulsive effect, the point B_2' is mapped to a point B_3 which is in the phase set $x = h+\tau$, where $y_{M'} < y_{B_2'} < y_A$ and $y_M < y_{B_3} < y_{B_1}$.

Repeat the above steps, the trajectory from point B will come across impulsive effect infinitely times. Denote the phase point corresponding to the ith impulsive effect, which is also the order i successor point of point B as B_i, $i = 1, 2, \ldots$. Let $B_0 = B$, then we have

$$y_{B_0} < y_{B_2} < y_{B_4} < \cdots < y_{B_{2k}} < y_{B_{2(k+1)}} < \cdots < y_M,$$

and

$$y_{B_1} > y_{B_3} > y_{B_5} > \cdots > y_{B_{2k+1}} > y_{B_{2(k+1)+1}} > \cdots > y_M.$$

Thus $\{y_{B_{2k}}\}$, $k = 0, 1, 2, \ldots$, is a monotonically increasing sequence, and $\{y_{B_{2k+1}}\}$, $k = 0, 1, 2, \ldots$, is a monotonically decreasing sequence (see Fig. 4), and furthermore,

$$y_{B_{2k}} \to y_M, \quad \text{as } k \to \infty; \quad \text{and}$$
$$y_{B_{2k+1}} \to y_M, \quad \text{as } k \to \infty.$$

Choose an arbitrary point $Q_0 \in \overline{BB_1}$ different from the point M, Without loss of generality, we assume that $y_B < y_{Q_0} < y_M$ (otherwise, $y_M < y_{Q_0} < y_{B_1}$, the discussions are similar). There must exist an integer n_0 such that $y_{B_{2n_0}} < y_{Q_0} < y_{B_{2(n_0+1)}}$. The trajectory from point Q_0 will also undergo impulsive effect infinitely times. We denote the phase point corresponding to the kth impulsive effect as Q_k, $k = 1, 2, \ldots,$

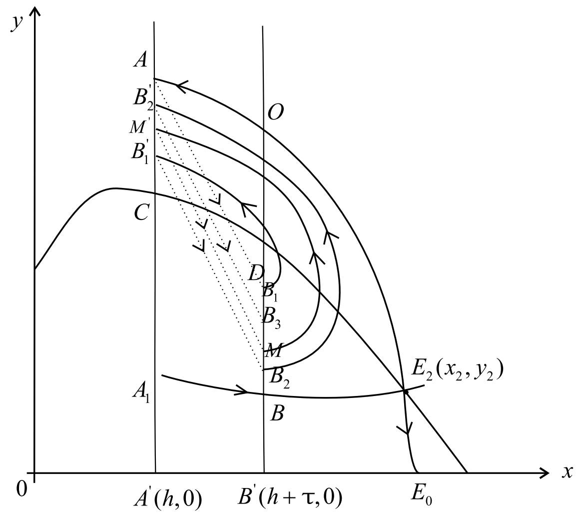

Fig. 4 The orbitally asymptotically stability of the order one periodic solution of the system (2)

then for any l, we have $y_{B_{2(n_0+l)}} < y_{Q_{2l}} < y_{B_{2(n_0+l+1)}}$ and $y_{B_{2(n_0+l+1)+1}} < y_{Q_{2l+1}} < y_{B_{2(n_0+l)+1}}$, so $\{y_{Q_{2l}}\}$, $l = 0, 1, 2, \ldots$, is also monotonically increasing, and $\{y_{Q_{2l+1}}\}$, $l = 0, 1, 2, \ldots$, is also monotonically decreasing, and

$$y_{Q_{2l}} \to y_M, \quad \text{as } k \to \infty; \quad \text{and}$$
$$y_{Q_{2l+1}} \to y_M, \quad \text{as } k \to \infty.$$

Therefore, in either case, the successor points of the phase points corresponding to the successive impulsive effect are attracted to point M, and thus the order one periodic solution of the system (2) is orbitally asymptotically stable. The proof is completed. □

Theorem 4 *Under the conditions of Theorem 3, the system (2) has no order k periodic solution in region Ω_1, where $k \geq 2$.*

Proof For any point $S \in \overline{B_1 O}$, $y_{B_1} < y_S < y_O$, the trajectory from point S will undergo impulsive effect infinitely times, and denote the impulse point and phase point corresponding to the kth impulsive effect as S_k' and S_k, respectively, where $k = 1, 2, \ldots$. It is easy to know $y_C < y_{S_k'} < y_A$ for $k = 1, 2, \ldots$, so we have $y_{S_k} = (1-\beta)y_{S_k'} < (1-\beta)y_A = y_{B_1} < y_S$, then the order k successor function $F_k(S) = y_{S_k} - y_S \neq 0$, which means there does not exist an order k periodic solution passing through point S, where $k = 1, 2, \ldots$.

For any point $S \in \overline{BM}$, $y_B < y_S < y_M$, there must exist an integer n_0 such that $y_{B_{2n_0}} < y_S < y_{B_{2(n_0+1)}}$. We denote the order k successor point of point S as point S_k, where $k = 1, 2, \ldots$. According to the proof of the Theorem 3, we have $y_{B_{2(n_0+l)}} < y_{S_{2l}} < y_{B_{2(n_0+l+1)}} < y_M$ and $y_M < y_{B_{2(n_0+l+1)+1}} < y_{S_{2l+1}} < y_{B_{2(n_0+l)+1}}$, so $\{y_{S_{2l}}\}, l = 0, 1, 2, \ldots$, is a monotonically increasing sequence where $S_0 = S$, and $\{y_{S_{2l+1}}\}, l = 0, 1, 2, \ldots$, is a monotonically decreasing sequence, and

$$y_{S_{2l}} \to y_M, \quad \text{as } k \to \infty; \quad \text{and}$$

$$y_{S_{2l+1}} \to y_M, \quad \text{as } k \to \infty.$$

When $k = 2l$, we have $y_{S_k} > y_{S_0} = y_S$ and the order k successor function $F_k(S) = y_{S_k} - y_S > 0$; when $k = 2l + 1$, we have $y_{S_k} > y_M > y_{S_0} = y_S$ and the order k successor function $F_k(S) = y_{S_k} - y_S > 0$. That is to say there does not exist an order k periodic solution passing through point S, where $k = 1, 2, \ldots$.

Analogously, for any point $S \in \overline{MB_1}$, $y_M < y_S < y_{B_1}$, we can prove there does not exist an order k periodic solution through point S, where $k = 1, 2, \ldots$.

For any point $S \in \overline{B'B}$, the trajectory of the system (2) from point S must ultimately pass through the x-axis and doesn't go through any impulsive effect.

From the above discussion, we know the system (2) has no order k periodic solution in region Ω_1, where $k \geq 2$. That completes the proof. □

Theorem 5 *Under the conditions of Theorem 2, if $\beta = \beta^*$, then the system (2) has an order one homoclinic cycle which is the unique order one circle in region Ω_1; if $\beta \in (\beta^*, 1)$, then the system (2) has no order one periodic solution in region Ω_1.*

Proof When $\beta = \beta^*$, we have $(1 - \beta)y_A = y_{B_1} = y_B$. It is easy to know that the curve $\widehat{BE_2OA} \cup \overline{AB}$ is an order one circle which has the saddle $E_2(x_2, y_2)$ in it. According to Theorem (2), we know it is the unique order one periodic solution of the system (2) in the region Ω_1, that is to say, the system (2) has an order one homoclinic cycle which is the unique order one circle in region Ω_1.

When $\beta \in (\beta^*, 1)$, we have $0 < (1 - \beta)y_A = y_{B_1} < y_B$, the trajectory of the system (2) from point B_1 passes through the x-axis and doesn't go through any impulsive effect. Beside, the trajectory of the system (2) from point $H \in \overline{BO}$ must pass through the x-axis after undergoing once impulsive effect and the trajectory of the system (2) from point $H \in \overline{B'B}$ must pass through the x-axis without undergoing any impulsive effect. So we get the result that the system (2) has no order one periodic solution in region Ω_1. That completes the proof. □

4 Numerical simulations and discussions

In this paper, we build a predator–prey model with both constant rate harvesting and state dependent impulsive harvesting. The constant term models the human behavior of the frequent predator harvesting, and the impulsive term models the human behavior of infrequent predator harvesting. Combination of the above two harvesting methods is commonly applied in practice, so our model is more realistic and can provide reliable tactic basis for the practical species management. In system (2), we assume the predator has high commercial value, besides a constant rate harvesting, we harvest the predator in pulses on the basis of the amount of the prey. Adopting two harvesting methods allows us to exploit the predator resources more fully without resource exhaustion.

Under the parametric conditions listing in the Theorem 1, we prove that the system (1) has two equilibria: a saddle $E_2(x_2, y_2)$ and a locally asymptotically stable node or focus $E_1(x_1, y_1)$, and system (1) has no closed orbits in the interior of the first quadrant. Every solution whose initial value is in the region bounded by the two stable flow of $E_2(x_2, y_2)$ tends to the equilibrium $E_1(x_1, y_1)$ as $t \to \infty$ (see Fig. 5). But other solutions will pass through the x-axis after limited time, which means the predator ultimately die out (see Fig. 6). This indicates that if we just carry out a constant rate harvesting and don't harvest the predator in pulses, we should control the amount of the predator and the prey within a specific scale otherwise the predator will die out finally.

When we consider an impulsive harvesting of the predator and add impulsive conditions to the system (1), we prove that the system (2) may exist an order one periodic solution, and the existence is mainly dependent on the level of impulsive harvesting β. Under the parametric conditions listing in Theorem 2, we prove that there must exist fixed values β^0 and β^* such that the system (2) exists a unique order one periodic solution in the region Ω_1 when the harvesting

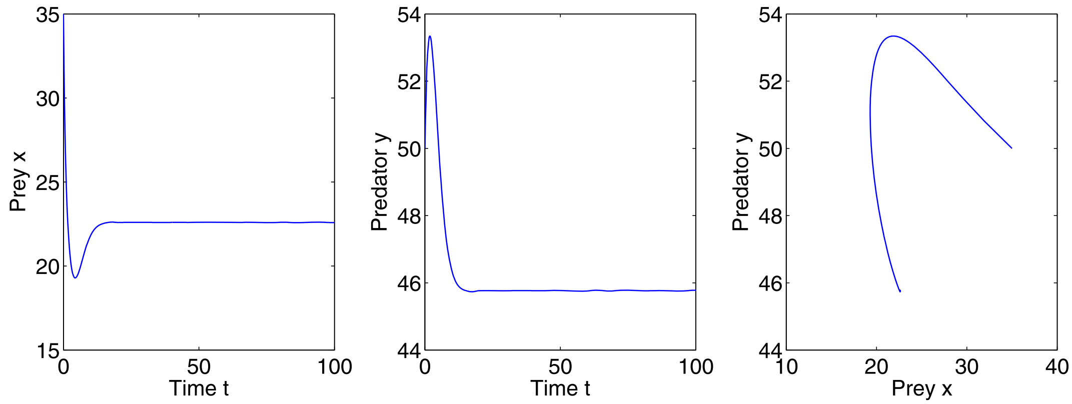

Fig. 5 The time series and the portrait phase of the system (1) when $r = 2$, $K = 40$, $a = 30$, $\mu = 0.8$, $D = 0.3$, $H = 2$ and $(x(0), y(0)) = (35, 50)$

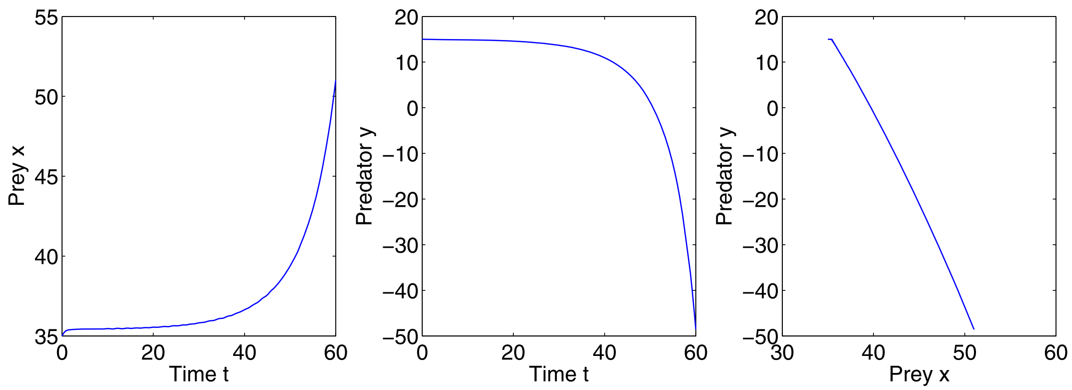

Fig. 6 The time series and the portrait phase of the system (1) when $r = 2$, $K = 40$, $a = 30$, $\mu = 0.8$, $D = 0.3$, $H = 2$ and $(x(0), y(0)) = (35, 15)$

rate is bounded below by β^0 and above by β^*, respectively. Under the parametric conditions of Theorem 3, we prove that the unique order one periodic solution is orbitally asymptotically stable. Within the allowed range, as the β increases, the period of the order one periodic solution becomes longer, and thus the amplitude of the order one periodic solution becomes larger (see Fig. 7). Besides, we also prove that the system (2) doesn't have order k ($k = 2, 3, \ldots$) periodic solutions in Theorem 4, then the order one periodic solution is the unique periodic solution. These results illustrate that if we can satisfy the parametric conditions in Theorem 3, we can increase the impulsive harvest yield as high as possible without worrying about resource exhaustion.

Like a lot of other predator–prey systems [23, 24], the system (2) also exhibits bifurcation phenomenon. According to the conclusions of Theorems 2, 3 and 5, we can choose the parameter β as a bifurcation parameter such that the impulsive differential equations (2) exhibits the phenomenon of homoclinic bifurcation. Similar to many ordinary differential equation systems, under the parametric conditions of Theorem 2, there exists a bifurcation point $\beta = \beta^*$ for the system (2). When $\beta = \beta^*$, the system (2) has an order one homoclinic cycle which is the unique order one circle in region Ω_1. When β is gradually changed from $\beta = \beta^*$ to $\beta^0 < \beta < \beta^*$, the order one homoclinic cycle is broken and a new order one periodic solution is generated at the same time. We also give the conditions that guarantee the unique order one periodic solution is orbitally asymptotically stable in Theorem 3. The order one periodic solution must change gradually from orbitally asymptotically stable to unstable as β gradually increase from β^0 to β^*. Besides, when β is gradually changed from $\beta = \beta^*$ to $1 > \beta > \beta^*$, the

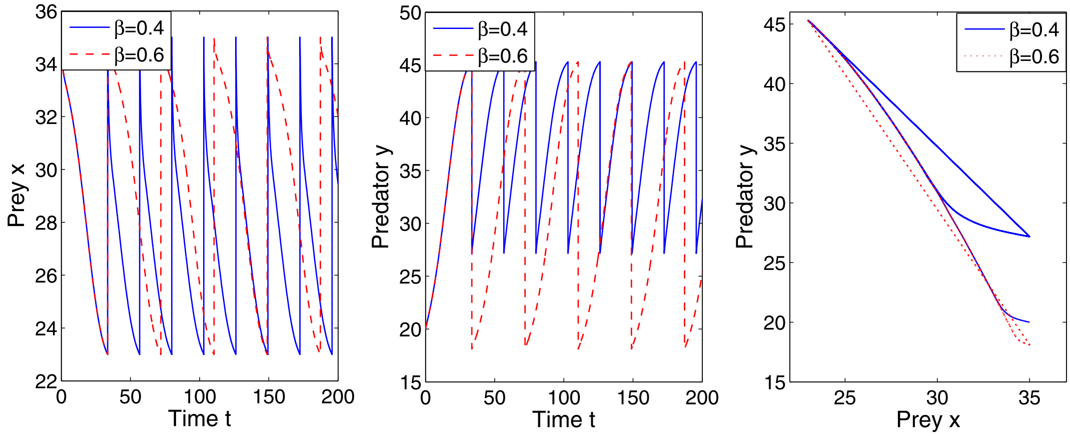

Fig. 7 The time series and the portrait phase of the system (2) when $r = 2$, $K = 40$, $a = 30$, $\mu = 0.8$, $D = 0.3$, $H = 2$, $h = 23$, $\tau = 12$ and $(x(0), y(0)) = (35, 20)$

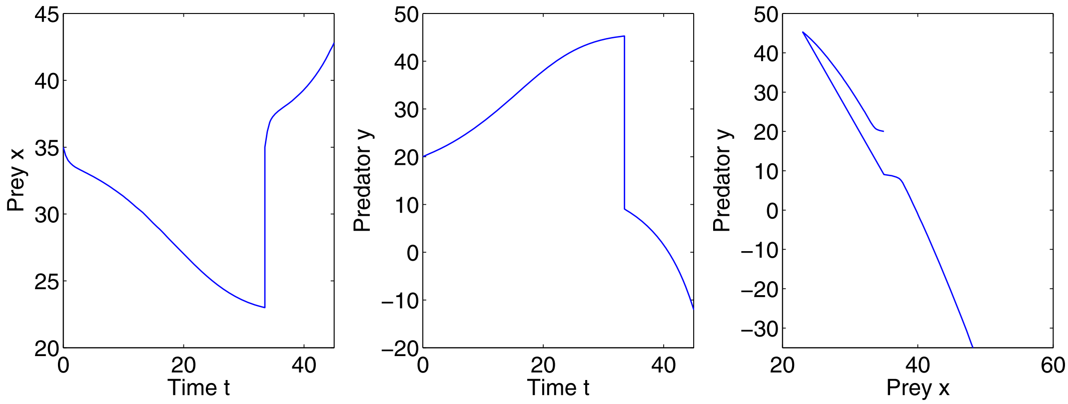

Fig. 8 Time series and portrait phase of the system (2) when $r = 2$, $K = 40$, $a = 30$, $\mu = 0.8$, $D = 0.3$, $H = 2$, $h = 23$, $\tau = 12$, $\beta = 0.8$ and $(x(0), y(0)) = (35, 20)$

order one homoclinic cycle is also broken, but no new order one periodic solution is generated at the same time, that is to say, the system (2) will have no periodic solution in the region Ω_1 and all the solutions passing through the region Ω_1 will reach the x-axis after limited time which means the predators will finally die out (see Fig. 8). These results illustrate that it also demands reasonable control of the impulsive harvest yield in order to form a good ecological environment and avoid the occurrence of resource exhaustion.

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