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Research Article

Periodic Solutions and Homoclinic Bifurcations of Two Predator-Prey Systems with Nonmonotonic Functional Response and Impulsive Harvesting

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Two predator-prey models with nonmonotonic functional response and state-dependent impulsive harvesting are formulated and analyzed. By using the geometry theory of semicontinuous dynamic system, we obtain the existence, uniqueness, and stability of the periodic solution and analyse the dynamic phenomenon of homoclinic bifurcation of the first system by choosing the harvesting rate β as control parameter. Besides, we also study the homoclinic bifurcation of the second system about parameter δ on the basis of the theory of rotated vector field. Finally, numerical simulations are presented to illustrate the results.

1. Introduction

Predator-prey interaction is one of the most important relationships in the ecosystem, so it has long been a focus of study in mathematical ecology. The functional response of the predator to the prey describes how the predator density changes with the prey density. The well-known response functions, which have appeared in a lot of literatures, include linear function, Holling type-II function, Holling type-III function, and so on. The above mentioned response functions are all monotonic, and they are really accurate to describe the predator-prey interactions in many cases. But in the recent decades, scholars in different fields found through experiments that nonmonotonic functional response occurs in some predator-prey interactions. For example, if the prey exhibits group defense [1, 2], the predator growth rate will be inhibited when the prey density reaches a high level. This phenomenon is known to exist widely in nature and considerable work on it has been studied [3–8]. To study the predator-prey interaction when the prey exhibits group defense, Ruan and Xiao in [4] proposed the following model:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{xy}{a+x^2}, \\ \frac{dy}{dt} &= y\left(\frac{\mu x}{a+x^2}-D\right), \end{aligned} \tag{1}$$

where x and y represent the population densities of prey and predator, respectively; $K > 0$ and $r > 0$ represent the carrying capacity and the intrinsic birth rate of the prey, respectively, and $D > 0$ is the death rate of the predator. The function $p(x) = x/(a + x^2)$ denotes the predator functional response which can model the phenomenon of group defense because there exists $\overline{h} > 0$ such that $p'(x) > 0$ for $0 \le x < \overline{h}$ and $p'(x) < 0$ for $x > \overline{h}$.

Harvesting strategy of biological resources is also a focus topic in mathematical bioeconomics because it relates to the optimal management of renewable resources. In many literatures, impulsive differential equations are used to model the human action of harvesting. Predator-prey systems with periodic impulsive harvesting have been studied extensively and important results have been achieved [9–12]. In recent years, more and more scholars begin to consider state dependent impulsive harvesting instead of the periodic impulsive form [13–17]. In point of ecology, we would better know some information about the amount of the species when we harvest them, thus we can avoid excessive exploitation and resource exhaustion. To this end, we introduce a reliable real-time monitoring system to estimate the number of the species. Such monitoring systems exist in many cases. According to the feedback information from the monitoring system, we can manage our resources better. For predator-prey system (1), we assume the predator has high commercial value for human beings, and people increase its production mainly through replenishing its prey. In such an ecosystem, we suppose the amount of the prey can be estimated by a monitoring system, and the monitoring data can help us to decide if we harvest the predator or not. To model this harvesting behavior, we can propose the following state-dependent impulsive differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{xy}{a+x^2}, \\ \frac{dy}{dt} &= y\left(\frac{\mu x}{a+x^2}-D\right), \\ & \qquad x > h, \\ \Delta x &= \tau, \\ \Delta y &= -\beta y, \\ & \qquad x = h, \end{aligned} \tag{2}$$

where $h > 0$ is a threshold. When the amount of the prey is large than h, which means the food of the predator is abundant, and the development of the system coincide with our economic interest. When the amount of the prey drops to the threshold h, which means the nutrition of the predator will be deficient, we harvest the predator at rate $\beta \in (0, 1)$ and replenish some prey at the same time. We denote the recruitment of the prey as τ.

A lot of research has shown that the population size of the predator is influenced not only by the prey populations but also by the relative rate of prey population growth. If we take the relative growth rate effect into consideration, the above model needs some changes. To this end, we propose the following system:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{xy}{a+x^2}, \\ \frac{dy}{dt} &= y\left(\frac{\mu x}{a+x^2}-D\right)+\delta\left[r\left(1-\frac{x}{K}\right)-\frac{y}{a+x^2}\right]^2, \\ & \qquad x > h, \\ \Delta x &= \tau, \\ \Delta y &= -\beta y, \\ & \qquad x = h, \end{aligned} \tag{3}$$

where the term $\delta[r(1-(x/K))-y/(a+x^2)]^2$ represents the effect of the prey relative growth rate on the predator.

In this paper, we mainly discuss the dynamics properties of the systems (2) and (3). The paper is organized as follows. In Section 2, some notations and definitions of the geometric theory of semicontinuous dynamical systems are provided. In Section 3, by using the geometry theory of semicontinuous dynamic system, we firstly study the existence, uniqueness, and orbital stability of periodic solutions and analyse the dynamic phenomenon of homoclinic bifurcation of system (2) by choosing the harvesting rate β as control parameter. Then, we study the homoclinic bifurcation of system (3) about parameter δ on the basis of the theory of rotated vector field. The paper ends with a brief discussion and some numerical simulations.

2. Preliminaries

In this section, we give some notations and definitions of the geometric theory of semicontinuous dynamical systems which will be useful for the following discussion.

Definition 1 (see [9]). Consider the state-dependent impulsive differential equations

$$\begin{aligned} &\frac{dx}{dt} = \overline{P}(x,y), \quad \frac{dy}{dt} = \overline{Q}(x,y), \quad (x,y) \notin M\{x,y\}, \\ &\Delta x = \alpha(x,y), \quad \Delta y = \beta(x,y), \quad (x,y) \in M\{x,y\}. \end{aligned} \tag{4}$$

We define the dynamic system consisting of the solution mapping of the system (4) a semicontinuous dynamical system, denoted as (Ω, f, φ, M). We require that the initial point P of the system (4) should not be in the set $M\{x,y\}$, that is, $P \in \Omega = R_+^2 \setminus M\{x,y\}$, and the function φ is a continuous mapping that satisfies $\varphi(M) = N$. Here φ is called the impulse mapping, where $M\{x,y\}$ and $N\{x,y\}$ represent the straight lines or curves in the plane R_+^2, $M\{x,y\}$ is called the impulse set, and $N\{x,y\}$ is called the phase set.

Remark 2. For the systems (2) and (3), $M = \{(x,y) \mid x = h, y \geq 0\}$, $N = \{(x,y) \mid x = h+\tau, y \geq 0\}$, and for any $(x,y) \in M$, we have $\varphi(x,y) = (h+\tau, (1-\beta)y)$.

Definition 3 (see [9]). For the semicontinuous dynamical system defined by the state-dependent impulsive differential equations (4), the solution mapping $f(P,t) : \Omega \rightarrow \Omega$ consists of two parts.

(1) Let $\pi(P,t)$ denote the poincaré mapping with the initial point P of the following system:

$$\begin{aligned} \frac{dx}{dt} &= \overline{P}(x,y), \\ \frac{dy}{dt} &= \overline{Q}(x,y). \end{aligned} \tag{5}$$

If $f(P,t) \cap M\{x,y\} = \emptyset$, then $f(P,t) = \pi(P,t)$.

(2) If there exists a time point T_1 such that $f(P,T_1) = H \in M\{x,y\}$, $\varphi(H) = \varphi(f(P,T_1)) = P_1 \in N\{x,y\}$, and $f(P_1,t) \cap M\{x,y\} = \emptyset$, then $f(P,t) = \pi(P,T_1) + f(P_1,t)$ (see Figure 1(a)).

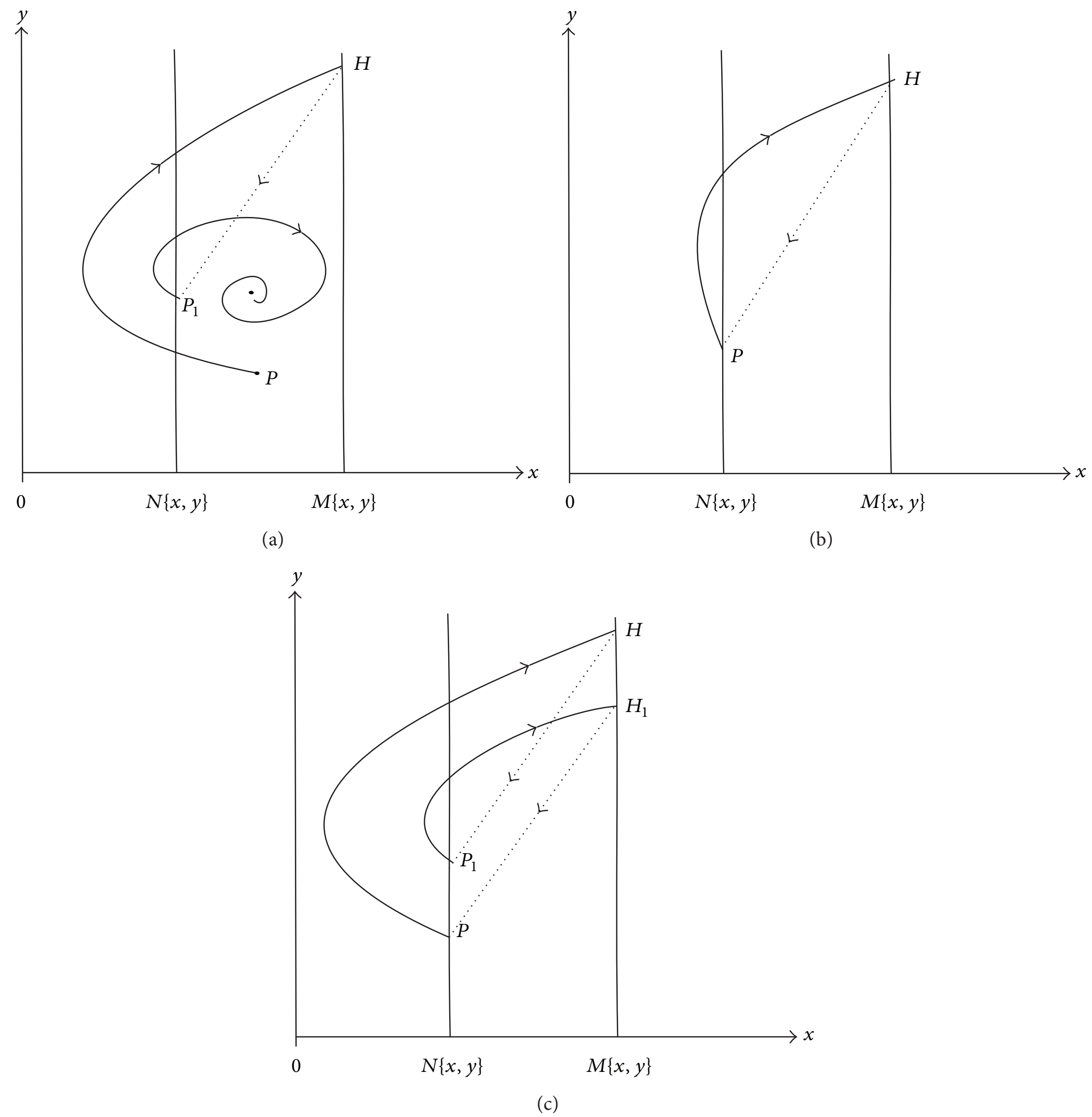

Figure 1: (a) The solution mapping of the system (4). (b) The order one periodic solution of the system (4). (c) The order two periodic solution of the system (4).

Remark 4. For (2) in Definition 3, if $f(P_1, t) \cap M\{x, y\} \neq \emptyset$, and there exists a time point T_2 such that $f(P_1, T_2) = H_1 \in M\{x, y\}$, $\varphi(H_1) = \varphi(f(P_1, T_2)) = P_2 \in N\{x, y\}$, and $f(P_2, t) \cap M\{x, y\} = \emptyset$, then $f(P, t) = \pi(P, T_1) + f(P_1, t) = \pi(P, T_1) + \pi(P_1, T_2) + f(P_2, t)$.

If $f(P_2, t) \cap M\{x, y\} \neq \emptyset, \ldots, f(P_{k-1}, t) \cap M\{x, y\} \neq \emptyset$ and $f(P_k, t) \cap M\{x, y\} = \emptyset$, then we can repeat the above steps and have the following form:

$$f(P, t) = \sum_{i=1}^{k} \pi(P_{i-1}, T_i) + f(P_k, t), \quad P_0 = P. \tag{6}$$

Definition 5 (see [9]). If there exists a point $P \in N\{x, y\}$ and a time point T_1 such that $f(P, T_1) = H \in M\{x, y\}$ and $\varphi(H) = \varphi(f(P, T_1)) = P \in N\{x, y\}$, then $f(P, t)$ is called an order one periodic solution of the system (4) whose period is T_1 (see Figure 1(b)). The orbit of the order one periodic solution is called an order one cycle. If there exists a singularity in the order one cycle, we call it an order one singular cycle. If the singularity is a saddle, we call it an order one homoclinic cycle.

Definition 6 (see [9]). If there exists a point $P \in N\{x, y\}$ and a time point T_1 such that $f(P, T_1) = H \in M\{x, y\}$, $\varphi(H) = P_1 \in N\{x, y\}$ and $P \neq P_1$, and there also exists a time point T_2 such that $f(P_1, T_2) = H_1 \in M\{x, y\}$ and $\varphi(H_1) = P \in N\{x, y\}$, then $f(P, t)$ is called an order two periodic solution of the system (4) whose period is $T_1 + T_2$ (see Figure 1(c)). Analogously, we can define the order k periodic solution of the system (4).

Definition 7. Suppose $\Gamma = f(P, t)$ is an order one periodic solution of the system (4). If for any $\varepsilon > 0$, there must exist $\delta > 0$ and $t_0 \geq 0$, such that for any point $P_1 \in U(P, \delta) \cap N\{x, y\}$, we have $\rho(f(P_1, t), \Gamma) < \varepsilon$ for $t > t_0$; then we call the order one periodic solution Γ orbitally asymptotically stable.

Definition 8 (see [9]). Suppose the impulse set M and phase set N of the system (4) are straight lines and a coordinate system can be defined in the phase set N. Let point $A \in N$ and its coordinate is a. Assume that the trajectory from the point A intersects the impulse set M at a point A', and, after impulsive effect, the point A' is mapped to the point $A_1 \in N$ with the coordinate a_1; then we call point A_1 the order one successor point of point A, and the order one successor function of point A is $F_1(A) = a_1 - a$.

Remark 9. For Definition 8, if the trajectory from the point A_1 intersects the impulse set M again at a point A'_1, and, after impulsive effect, the point A'_1 is mapped to the point $A_2 \in N$ with the coordinate a_2, then the point A_2 is obviously the order one successor point of point A_1; we also call point A_2 the order two successor point of point A, and the order two successor function of point A is $F_2(A) = a_2 - a$. If the process can be repeated over and over again, then we can define the order k successor point of point A (which we denote as A_k and its coordinate is a_k) and the order k successor function of A which we denote as $F_k(A) = a_k - a$.

Remark 10. For the systems (2) and (3), for any point $H \in N$, we define the directed distance between point H and the x-axis as the coordinate of point H. In this paper, we denote the coordinate of point H as y_H.

Lemma 11 (see [9]). *Successor function $F_k(A)$ is continuous.*

Lemma 12. *For the systems* (2) *and* (3)*, if there exists two points $A \in N$, $B \in N$ such that $F_1(A)F_1(B) < 0$, then there must exist a point $C \in N$ which is between the points A and B such that $F_1(C) = 0$; thus, the system must have an order one periodic solution which passes through the point C.*

Proof. By Lemma 11, we can easily get that there must exist a point $C \in N$ which is between the points A and B such that $F_1(C) = 0$. According to Definition 6, we know $\Gamma = f(C, t)$ is an order one periodic solution. That completes the proof. □

Lemma 13. *For the systems* (2) *and* (3)*, suppose point $A \in N$ and $F_k(A) \neq 0$, then the system does not have an order k periodic solution which passes through the point A.*

3. Periodic Solution and Homoclinic Bifurcation

In this section, we firstly discuss the existence, uniqueness, and stability of the periodic solution of the system (2) by using differential equation geometry theory then study the dynamic phenomena of homoclinic bifurcation of both systems (2) and (3). We now list some results about the system (1) that are given by Ruan and Xiao in [4].

For the system (1), there is always a hyperbolic saddle point at the origin $(0, 0)$ and an equilibrium $(K, 0)$ in the x-axis. When $\mu^2 - 4aD^2 > 0$, the system (1) may have one or two positive equilibria. If the positive equilibria exist, we denote them as (x_1, y_1), (x_2, y_2), where

$$\begin{aligned} x_1 &= \frac{\mu - \sqrt{\mu^2 - 4aD^2}}{2D}, \qquad y_1 = r\left(1 - \frac{x_1}{K}\right)\left(a + x_1^2\right), \\ x_2 &= \frac{\mu + \sqrt{\mu^2 - 4aD^2}}{2D}, \qquad y_2 = r\left(1 - \frac{x_2}{K}\right)\left(a + x_2^2\right). \end{aligned} \tag{7}$$

Now, we give some results appeared in [4].

Lemma 14 (see [4]). *If $4aD^2 < \mu^2 \leq (16/3)aD^2$ and $x_1 < K < x_2$ (or if $\mu^2 > (16/3)aD^2$ and $x_1 < K < -x_1 + 2\sqrt{\mu x_1/D}$), then the system* (1) *has three equilibria: two hyperbolic saddles $(0, 0)$ and $(K, 0)$ and a globally asymptotically stable equilibrium (x_1, y_1) in the interior of the first quadrant.*

Lemma 15 (see [4]). *If $4aD^2 < \mu^2 < (16/3)aD^2$ and $x_2 < K < (2\mu - \sqrt{\mu^2 - 4aD^2})/2D$ (or if $\mu^2 > 4aD^2$ and $K > (2\mu + \sqrt{\mu^2 - 4aD^2})/2D$), then the system* (1) *has four equilibria: two hyperbolic saddles $(0, 0)$ and (x_2, y_2), a hyperbolic stable node $(K, 0)$, and a stable (an unstable) equilibrium (x_1, y_1), and system* (1) *has no closed orbits.*

3.1. Periodic Solution and Homoclinic Bifurcation of System (2) *about Parameter β.* According to the model (2), the threshold h and the recruitment τ of the prey should satisfy the condition $0 < h < h + \tau < K$ by ecological significance. For this consideration, we have the following results.

Theorem 16. *If $4aD^2 < \mu^2 \leq (16/3)aD^2$, $x_1 < K < x_2$ (or if $\mu^2 > (16/3)aD^2$, $x_1 < K < -x_1 + 2\sqrt{\mu x_1/D}$), and $x_1 < h < h + \tau < K$, then there must exist a fixed value $\beta^0 \in (0, 1)$ such that for every $\beta \in (\beta^0, 1)$, the system* (2) *has a unique order one periodic solution in region Ω_1, where region Ω_1 is the region enclosed by the x-axis, the impulse set $x = h$, and the unstable flow of the saddle $(K, 0)$.*

Proof. According to Lemma 14, the system (1) has three equilibria: two hyperbolic saddles $(0, 0)$ and $(K, 0)$ and a globally asymptotically stable equilibrium (x_1, y_1) in the interior of the first quadrant. For convenience, we suppose the x-axis intersects the impulse set $x = h$ and the phase set $x = h + \tau$ at point A' and point B, respectively. The unstable flow of $(K, 0)$ also passes through the impulse set and the phase set, we denote the intersections by point A and point O, respectively. Besides, we suppose the vertical isocline $dx/dt = 0$ intersects the impulse set and the phase set at point C and point D, respectively. Then the region Ω_1 is the interior of the closed curve $\widehat{E_0OA} \cup \widehat{ACA'} \cup \widehat{A'BE_0}$, where E_0 denotes the equilibrium $(K, 0)$ (see Figure 2).

According to the impulsive conditions of the system (2), there must exist a fixed value $\beta^0 \in (0, 1)$, when $\beta = \beta^0$, point

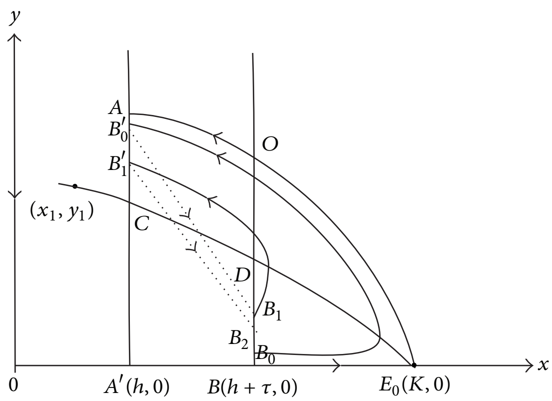

FIGURE 2: Existence of order one periodic solution of (2).

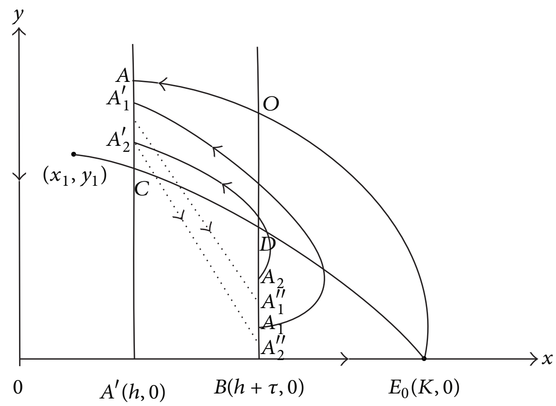

FIGURE 3: Uniqueness of order one periodic solution of (2).

A is mapped to the point D after impulsive effect, that is to say, $(1-\beta^0)y_A = y_D$.

When $\beta^0 < \beta < 1$, after impulsive effect, point A is mapped to a point between points B and D. Select a point B_0 next to B in the phase set $x = h+\tau$, the trajectory of the system (2) from point B_0 must intersect the impulse set $x = h$ at a point B_0' which is next to point A, and after impulsive effect, point B_0' is mapped to a point B_1 which is the order one successor point of B_0; then, we have $y_{B_1} = (1-\beta)y_{B_0'} < (1-\beta)y_A < y_D$. That is to say point B_1 is between points D and B_0. The trajectory from point B_1 must intersect the impulse set $x = h$ again at a point B_1', and after impulsive effect, point B_1' is mapped to a point B_2. Since distinct trajectories do not intersect, we can easily have $y_C < y_{B_1'} < y_{B_0'}$ and $y_{B_2} = (1-\beta)y_{B_1'} < (1-\beta)y_{B_0'} = y_{B_1}$. Obviously, point B_2 is the order one successor point of point B_1, and then we have the following results of the order one successor function:

$$\begin{aligned} F_1(B_0) &= y_{B_1} - y_{B_0} > 0, \\ F_1(B_1) &= y_{B_2} - y_{B_1} < 0. \end{aligned} \tag{8}$$

By Lemma 12, we know that there must exist a point M in the phase set $x = h+\tau$ which is between the points B_0 and B_1 such that $F_1(M) = 0$. Then we know the system (2) has an order one periodic solution which passes through the point M.

In the following, we prove the uniqueness of the order one periodic solution. Arbitrarily choose two points A_1 and A_2 in the phase set $x = h+\tau$, where $0 \le y_{A_1} < y_{A_2} \le y_D$. The trajectories of the system (2) through points A_1 and A_2 must intersect the impulse set $x = h$ at some points A_1' and A_2', respectively, and after impulsive effect, the points A_1' and A_2' must be mapped to the phase set $x = h+\tau$ at some points A_1'' and A_2'', respectively (see Figure 3). Because distinct trajectories do not intersect, we can easily have $y_C < y_{A_2'} < y_{A_1'} < y_A$ and $y_{A_2''} = (1-\beta)y_{A_2'} < y_{A_1''} = (1-\beta)y_{A_1'}$; then we get the order one successor functions must satisfy

$$\begin{aligned} F_1(A_2) - F_1(A_1) &= \left(y_{A_2''} - y_{A_2}\right) - \left(y_{A_1''} - y_{A_1}\right) \\ &= \left(y_{A_2''} - y_{A_1''}\right) + \left(y_{A_1} - y_{A_2}\right) < 0, \end{aligned} \tag{9}$$

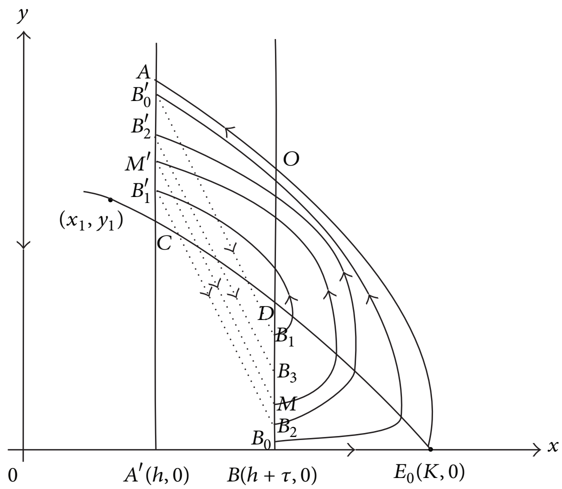

FIGURE 4: Illustration of the orbital asymptotic stability of the order one periodic solution of the system (2).

which means the order one successor function F_1 is monotone decreasing in the line segment $\overline{BD}$, thus there exists only one point $M \in \overline{BD}$ such that $F_1(M) = 0$.

Besides, for any point $H \in \overline{DO}$, it is easy to know that $F_1(H) < 0$. So the system (2) has only one order one periodic solution in the region Ω_1. That completes the proof. □

Theorem 17. *Under the conditions of Theorem 16, the order one periodic solution of the system* (2) *is orbitally asymptotically stable.*

Proof. According to Theorem 16, the system (2) has a unique order one periodic solution that passes through the point M which is in the phase set $x = h+\tau$, where $y_{B_0} < y_M < y_{B_1}$. The trajectory of the system (2) through point B_1 intersects the impulse set $x = h$ at point B_1', and after impulsive effect, the point B_1' is mapped to point B_2, where $y_B < y_{B_2} < y_M$. Besides, the trajectory of the system (2) from point B_2 must intersect the impulse set $x = h$ again at a point B_2', and after impulsive effect, the point B_2' is mapped to a point B_3 in the phase set $x = h+\tau$, where $y_{B_1} > y_{B_3} > y_M$ (see Figure 4).

Repeat the above steps, the trajectory from point B_0 will come across impulsive effect infinite times. Denote the phase point corresponding to the ith impulsive effect by B_i, $i = 1, 2, \ldots$. Obviously, B_i is also the order i successor point of point B_0. Then we have

$$y_{B_0} < y_{B_2} < y_{B_4} < \cdots < y_{B_{2k}} < y_{B_{2(k+1)}} < \cdots < y_M,$$
$$y_{B_1} > y_{B_3} > y_{B_5} > \cdots > y_{B_{2k+1}} > y_{B_{2(k+1)+1}} > \cdots > y_M. \tag{10}$$

Thus $\{y_{B_{2k}}\}$, $k = 0, 1, 2, \ldots$ is a monotonically increasing sequence, and $\{y_{B_{2k+1}}\}$, $k = 0, 1, 2, \ldots$ is a monotonically decreasing sequence. Because the order one successor function F_1 is monotone decreasing in the line segment $\overline{BD}$, we have

$$F_1(B_0) = y_{B_1} - y_{B_0} > F_1(B_2) = y_{B_3} - y_{B_2} > F_1(B_4)$$
$$= y_{B_5} - y_{B_4} > \cdots > 0, \tag{11}$$

and furthermore,

$$y_{B_{2k}} \longrightarrow y_M, \quad \text{as } k \longrightarrow \infty;$$
$$y_{B_{2k+1}} \longrightarrow I_M, \quad \text{as } k \longrightarrow \infty. \tag{12}$$

Pick any point $Q_0 \in \overline{B_0B_1}$ different from the point M; without loss of generality, we assume $y_{B_0} < y_{Q_0} < y_M$ (otherwise, $y_M < y_{Q_0} < y_{B_1}$; the discussions are similar), there must exist a natural number n_0 such that $y_{B_{2n_0}} < y_{Q_0} < y_{B_{2(n_0+1)}}$. The trajectory from point Q_0 will also undergo impulsive effect infinite times, and we denote the phase point corresponding to the kth impulsive effect as Q_k, $k = 1, 2, \ldots$. For any natural number l, we have $y_{B_{2(n_0+l)}} < y_{Q_{2l}} < y_{B_{2(n_0+l+1)}}$ and $y_{B_{2(n_0+l+1)+1}} < y_{Q_{2l+1}} < y_{B_{2(n_0+l)+1}}$. Then $\{y_{Q_{2l}}\}$, $l = 0, 1, 2, \ldots$ is also a monotonically increasing sequence, $\{y_{Q_{2l+1}}\}$, $l = 0, 1, 2, \ldots$ is also a monotonically decreasing sequence, and we get

$$y_{Q_{2l}} \longrightarrow y_M, \quad \text{as } l \longrightarrow \infty;$$
$$y_{Q_{2l+1}} \longrightarrow I_M, \quad \text{as } l \longrightarrow \infty. \tag{13}$$

This indicates that the trajectory of the system (2) from point $Q_0 \in \overline{B_0B_1}$ ultimately tends to the trajectory passing through point M, which means the order one periodic solution is orbitally asymptotically stable. That completes the proof. □

Theorem 18. *Under the conditions of Theorem 16, the system (2) has no order k periodic solution in region Ω_1, where $k \geq 2$.*

Proof. For any point $S \in \overline{B_1O}$, $y_{B_1} < y_S < y_O$, the trajectory from point S will undergo impulsive effect infinite times. Denote the impulsive point and phase point corresponding to the kth ($k = 1, 2, \ldots$) impulsive effect by S'_k and S_k, respectively. Similar to the discussions in Theorem 17, we can easily know $y_C < y_{S'_k} < y_{B'_0}$, $k = 1, 2, \ldots$, then we have $y_{S_k} = (1-\beta) y_{S'_k} < (1-\beta) y_{B'_0} = y_{B_1} < y_S$. Obviously, point S_k is the order k successor point of point S, then the order k successor function $F_k(S) = y_{S_k} - y_S \neq 0$. By Lemma 13, there does not exist an order k ($k = 1, 2, \ldots$) periodic solution through point S.

For any point $S \in \overline{BM}$, $y_B < y_S < y_M$, according to the proof of Theorem 17, there must exist a natural number n_0 such that $y_{B_{2n_0}} < y_S < y_{B_{2(n_0+1)}}$. Denote the order k ($k = 1, 2, \ldots$) successor point of point S by point S_k, then we have $y_{B_{2(n_0+l)}} < y_{S_{2l}} < y_{B_{2(n_0+l+1)}} < y_M$ and $y_M < y_{B_{2(n_0+l+1)+1}} < y_{S_{2l+1}} < y_{B_{2(n_0+l)+1}}$. Then $\{y_{S_{2l}}\}$, $l = 0, 1, 2, \ldots$ is amonotonically increasing sequence where $S_0 = S$, $\{y_{S_{2l+1}}\}$, $l = 0, 1, 2, \ldots$ is a monotonically decreasing sequence, and

$$y_{S_{2l}} \longrightarrow y_M, \quad \text{as } l \longrightarrow \infty;$$
$$y_{S_{2l+1}} \longrightarrow I_M, \quad \text{as } l \longrightarrow \infty. \tag{14}$$

When $k = 2l$, we have $y_M > y_{S_k} > y_{S_0} = y_S$ and the order k successor function $F_k(S) = y_{S_k} - y_S > 0$. When $k = 2l+1$, we have $y_{S_k} > y_M > y_{S_0} = y_S$ and $F_k(S) = y_{S_k} - y_S > 0$. That is to say there does not exist an order k ($k = 1, 2, \ldots$) periodic solution passing through point S.

Analogously, for any point $S \in \overline{MB_1}$, $y_M < y_S < y_{B_1}$, we can prove there does not exist an order k ($k = 1, 2, \ldots$) periodic solution passing through point S.

From the above discussion, we know that the order one periodic solution is the unique periodic solution of the system (2) and there is no order k ($k \geq 2$) periodic solution in region Ω_1. That completes the proof. □

Theorem 19. *If $4aD^2 < \mu^2 < (16/3)aD^2$, $x_2 < K < (2\mu - \sqrt{\mu^2 - 4aD^2})/2D$ (or if $\mu^2 > 4aD^2$, $K > (2\mu + \sqrt{\mu^2 - 4aD^2})/2D$), $x_1 < h < h + \tau < x_2$ and $K \leq \sqrt{3a}$, then there must exist two fixed values β^0 and β^* which satisfy $0 < \beta^0 < \beta^* < 1$ such that*

(1) *when $\beta = \beta^*$, the system (2) has an order one homoclinic cycle;*

(2) *if $\beta^0 < \beta < \beta^*$, then the homoclinic cycle of system (2) disappears and bifurcates an order one periodic solution in region Ω_2, where region Ω_2 is the region enclosed by the x-axis, the impulse set $x = h$, and the unstable flow of the saddle (x_2, y_2). Furthermore, the order one periodic solution is unique and is orbitally asymptotically stable;*

(3) *if $\beta^* < \beta < 1$, then the homoclinic cycle of system (2) also disappears and there is no periodic solution in region Ω_2.*

Proof. For $K \leq \sqrt{3a}$, it is easy to know that $y_1 > y_2$. According to Lemma 15, the system (1) has four equilibria: two hyperbolic saddles $(0, 0)$ and (x_2, y_2), a hyperbolic stable node $(K, 0)$, and a node or focus (x_1, y_1). Obviously, the x-axis, the vertical isocline $dx/dt = 0$, and the unstable and stable flow of (x_2, y_2) are all intersected with the impulse set

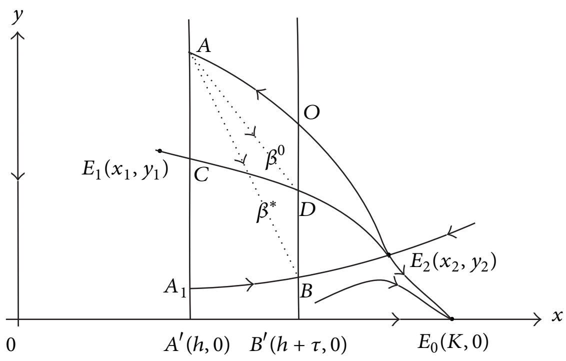

Figure 5: Illustration of homoclinic bifurcation of system (2) by choosing the harvesting rate β as control parameter.

$x = h$ and phase set $x = h + \tau$. For convenience, we denote the intersections by points A' and B', C and D, A and O, and A_1 and B, respectively (see Figure 5). Then the region Ω_2 is the interior of the closed curve $\widehat{E_0E_2OA} \cup \overline{ACA'} \cup \overline{A'B'E_0}$, where E_0 and E_2 denote the equilibria $(K, 0)$ and (x_2, y_2), respectively. According to the impulsive conditions of the system (2), there must exist a fixed value $\beta^0 \in (0, 1)$, when $\beta = \beta^0$, point A is mapped to the point D after impulsive effect. Also, there must exist a fixed value $\beta^* \in (\beta^0, 1)$, when $\beta = \beta^*$, point A is mapped to the point B after impulsive effect. Thus we have $(1 - \beta^0)y_A = y_D$ and $(1 - \beta^*)y_A = y_B$.

When $\beta = \beta^*$, we have $(1 - \beta)y_A = y_B$; we know the curve $\widehat{BE_2OA} \cup \overline{AB}$ is an order one circle which has the saddle $E_2(x_2, y_2)$ in it; that is to say, the system (2) has an order one homoclinic cycle.

If $\beta \in (\beta^0, \beta^*)$, after impulsive effect, point A is mapped to a point between points B and D. Similar to the discussion of Theorems 16 and 17, we can prove that the system (2) has an order one periodic solution which passes through a point $M \in \overline{BD}$ and is orbitally asymptotically stable. Besides, for any point $H \in \overline{B'B}$, the trajectory of the system (2) from point H must ultimately tend to the equilibrium $(K, 0)$ and does not undergo any impulsive effect, that is to say the system (2) has no periodic solution passing through the point H; so we can know the system (2) has a unique order one periodic solution in the region Ω_2 which is orbitally asymptotically stable.

When $\beta \in (\beta^*, 1)$, we have $0 < (1 - \beta)y_A < y_B$. Then the trajectory of the system (2) through point $H \in \overline{OB}$ must tend to the equilibrium $(K, 0)$ after coming across once impulsive effect and the trajectory passing through point $H \in \overline{B'B}$ must tend to the equilibrium $(K, 0)$ without undergoing any impulsive effect. So we get that the system (2) has no periodic solution in region Ω_2. That completes the proof. □

3.2. Homoclinic Bifurcation of System (3) *about Parameter* δ. By above analysis, we know there exists a $\beta^* \in (0, 1)$ such that system (2) has an order one homoclinic cycle and when the parameter β is appropriately changed, the homoclinic cycle disappears and bifurcates a unique stable order one periodic solution. In the following, we will choose δ as a control parameter and study homoclinic bifurcation of system (3) by using the theory of rotated vector fields. For the sake of convenience, we give some properties about rotated vector fields.

The perturbed system of system (4) with parameter δ is as follows:

$$\frac{dx}{dt} = \overline{P}(x, y, \delta), \qquad \frac{dy}{dt} = \overline{Q}(x, y, \delta), \qquad (x, y) \notin M\{x, y\}, \tag{15}$$
$$\Delta x = \alpha(x, y), \quad \Delta y = \beta(x, y), \quad (x, y) \in M\{x, y\}.$$

Let

$$\Delta = \begin{vmatrix} \overline{P} & \overline{Q} \\ \dfrac{\partial \overline{P}}{\partial \delta} & \dfrac{\partial \overline{Q}}{\partial \delta} \end{vmatrix}, \tag{16}$$

then we have the following definitions.

Definition 20 (see [18]). For any point on the trajectory of system (15), if $\Delta > 0$, then system (15) constitutes positive rotated vector fields concerning the parameter δ; otherwise, if $\Delta < 0$, system (15) constitutes negative rotated vector fields.

Lemma 21 (see [18]). *In the positive (negative) rotated vector fields of system* (15)*, the rotated direction of vector fields is counterclockwise (clockwise) when parameter δ changes from $\delta = 0$ to $\delta > 0$.*

Theorem 22. *If $4aD^2 < \mu^2 < (16/3)aD^2$, $x_2 < K < (2\mu - \sqrt{\mu^2 - 4aD^2})/2D$ (or if $\mu^2 > 4aD^2$, $K > (2\mu + \sqrt{\mu^2 - 4aD^2})/2D$), $x_1 < h < h + \tau < x_2$, $K \le \sqrt{3a}$, $\beta = \beta^*$, $\delta > 0$, and $\delta \ll 1$, then the system* (3) *has a unique order one periodic solution. Furthermore, it is orbitally asymptotically stable.*

Proof. From the discussion in Theorem 19, we know that when $\beta = \beta^*$ and $\delta = 0$, the system (3) has an order one homoclinic circle $\widehat{BE_2OA} \cup \overline{AB}$.

For the system (3), we denote $P(x, y) = rx(1 - (x/K)) - xy/(a + x^2)$ and $Q(x, y) = y(\mu x/(a + x^2) - D) + \delta[r(1 - (x/K)) - y/(a + x^2)]^2$. By simple calculation, we have

$$\Delta = \begin{vmatrix} P & Q \\ \dfrac{\partial P}{\partial \delta} & \dfrac{\partial Q}{\partial \delta} \end{vmatrix} = x\left[r\left(1 - \frac{x}{K}\right) - \frac{y}{a + x^2}\right]^3. \tag{17}$$

Divided by the vertical isocline $[r(1 - (x/K)) - y/(a + x^2)] = 0$, the rotated direction of vector fields below the vertical isocline is counterclockwise, but above the vertical isocline, the rotated direction of vector fields of system (3) is clockwise. When the parameter δ changes from 0 (the value in which heteroclinic cycle exists) to $0 < \delta_1 \ll 1$, both the unstable and stable flow of (x_2, y_2) move away from their original

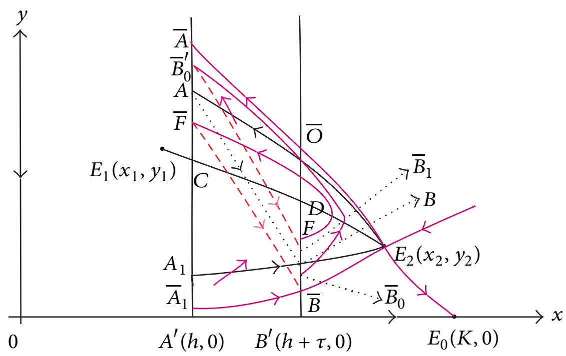

FIGURE 6: Illustration of homoclinic bifurcation of system (3) about δ.

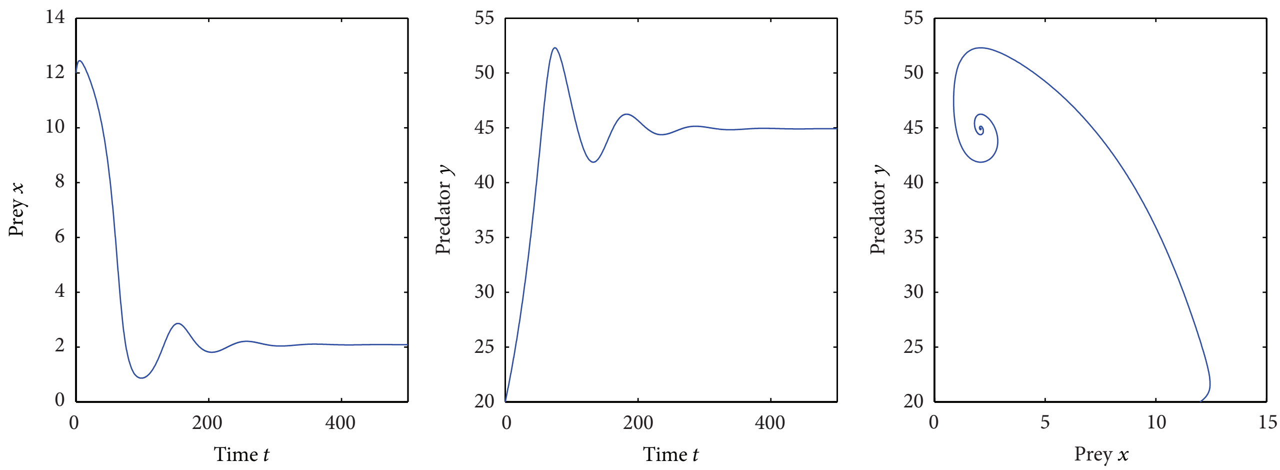

FIGURE 7: The time series and the portrait phase of the system (1) when $r = 0.5$, $K = 15$, $a = 100$, $\mu = 0.5$, $D = 0.01$, and $(x(0), y(0)) = (12, 20)$.

positions. We denote the new intersections of the unstable and stable flow of (x_2, y_2) and the impulse set $x = h$ and phase set $x = h + \tau$ are $\overline{A}$ and $\overline{O}$, and $\overline{A}_1$ and $\overline{B}$, respectively (see Figure 6). According to Theorem 19, we have $(1-\beta^*)y_A = y_B$; then there must exist a point $\overline{F}$ which is in the impulse set $x = h$ and below the point A such that $(1-\beta^*)y_{\overline{F}} = y_{\overline{B}}$. Obviously, system (3) must have a trajectory passing through the point $\overline{F}$, and we suppose it intersects with the phase set $x = h + \tau$ at point F.

Obviously, point $\overline{B}$ is the order one successor point of point F, then the order one successor function $F_1(F) = y_{\overline{B}} - y_F < 0$. Besides, select a point $\overline{B}_0$ next to $\overline{B}$ in the phase set $x = h + \tau$; the trajectory of the system (3) from point $\overline{B}_0$ must intersect the impulse set $x = h$ at a point $\overline{B}_0'$ which is next to point $\overline{A}$ and above point A, and after impulsive effect, point $\overline{B}_0'$ is mapped to a point $\overline{B}_1$ which is the order one successor point of $\overline{B}_0$, then we have $y_{\overline{B}_1} = (1-\beta^*)y_{\overline{B}_0'} > (1-\beta^*)y_A = y_B$. That is to say point $\overline{B}_1$ is above point B. Then we have the successor function $F_1(\overline{B}_0) = y_{\overline{B}_1} - y_{B_0} > 0$. By Lemma 12, we know that there must exist a point M in the phase set $x = h + \tau$ which is between the points $\overline{B}$ and F such that $F_1(M) = 0$. Then we know the system (3) has an order one periodic solution which passes through the point M.

Similar to the discussion of Theorems 16 and 17, we can prove the uniqueness and stability of the order one periodic solution. That completes the proof. □

4. Numerical Simulations and Discussions

In this paper, we have proposed two predator-prey models with nonmonotonic functional response and state dependent impulsive harvesting. If we take no account of the predator harvesting, Ruan and Xiao (2001) showed that the system (1) has no periodic solution under the conditions listing in the Theorem 16. Our numerical simulations also confirm such conclusions (see Figure 7). However, under the same conditions, if we further consider an impulsive predator harvesting, we get that the system (2) has a periodic solution in some cases. We also showed that the existence is mainly dependent on the harvesting rate β. We found that there exists a threshold $\beta^0 \in (0, 1)$, and the system must have a unique orbitally asymptotically stable order one periodic solution when the harvesting rate β is higher than β^0. When

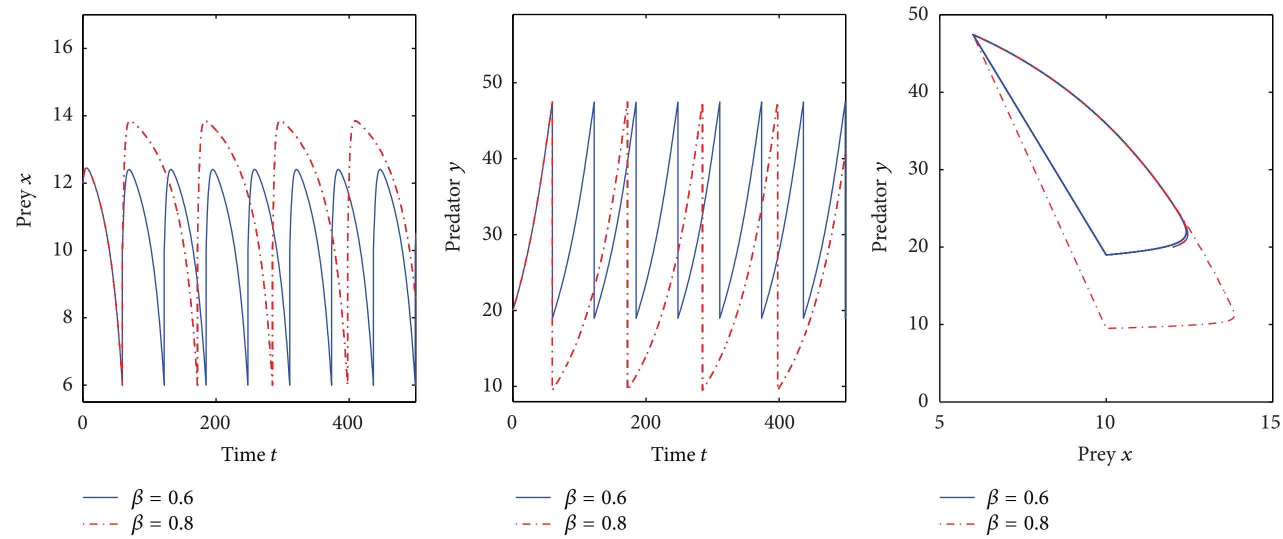

FIGURE 8: The time series and the portrait phase of the system (2) when $r = 0.5$, $K = 15$, $a = 100$, $\mu = 0.5$, $D = 0.01$, $h = 6$, $\tau = 4$, and $(x(0), y(0)) = (12, 20)$.

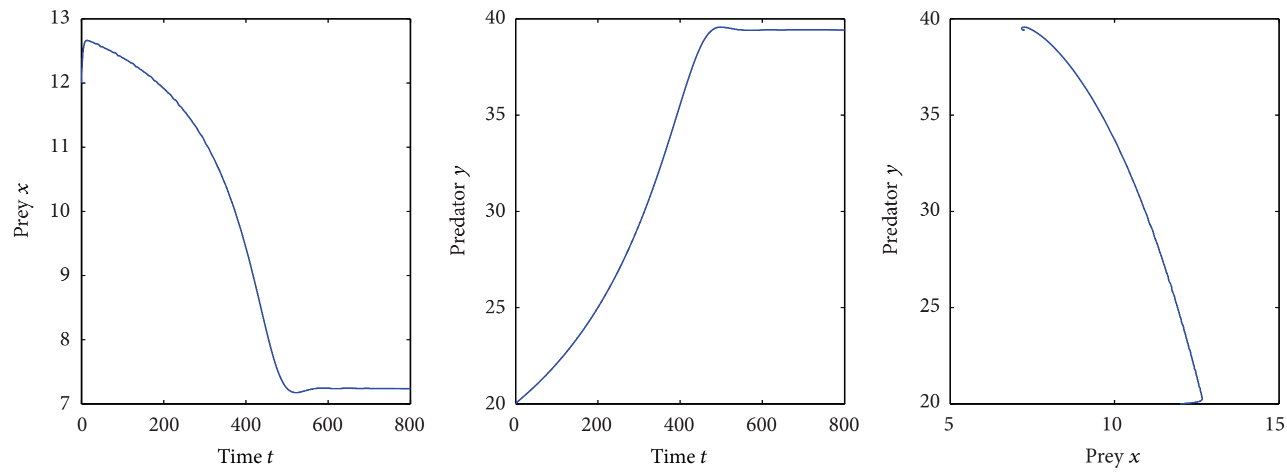

FIGURE 9: The time series and the portrait phase of the system (1) when $r = 0.5$, $K = 15$, $a = 100$, $\mu = 0.8$, $D = 0.038$, and $(x(0), y(0)) = (12, 20)$.

the periodic solution exists, our numerical simulations show that, with a bigger harvesting rate, the solution has a longer period and a larger amplitude (see Figure 8).

Besides, we also prove that the system (2) does not have order k, $(k = 2, 3, \ldots)$ periodic solutions in Theorem 18; then the order one periodic solution is the unique periodic solution. These results illustrate that if we can satisfy the conditions listing in Theorem 16, we can not only form a sustainable ecological system but also obtain large amounts of valuable biological resources.

Under the conditions listing in Theorem 19, the system (1) also does not have any periodic solution (see Figure 9) and we also give the conditions under which the system (2) has an order one periodic solution. Besides, according to the conclusions of Theorem 19, we can choose the parameter β as a bifurcation parameter such that the impulsive differential equation system (2) exhibits the phenomenon of homoclinic bifurcation. Similar to many ordinary differential equation systems, we can see that there exists a bifurcation point $\beta = \beta^*$ for the system (2). When $\beta = \beta^*$, the system (2) has an order one homoclinic cycle which is the unique order one circle in region Ω_2. When β gradually changes from $\beta = \beta^*$ to $\beta^0 < \beta < \beta^*$, the order one homoclinic cycle is broken and a new order one periodic solution which is orbitally asymptotically stable is generated (see Figure 10). When β gradually changes from $\beta = \beta^*$ to $\beta > \beta^*$, the order one homoclinic cycle is also broken but no new order one periodic solution is generated at the same time; that is to say, the system (2) will have no periodic solution (see Figure 11).

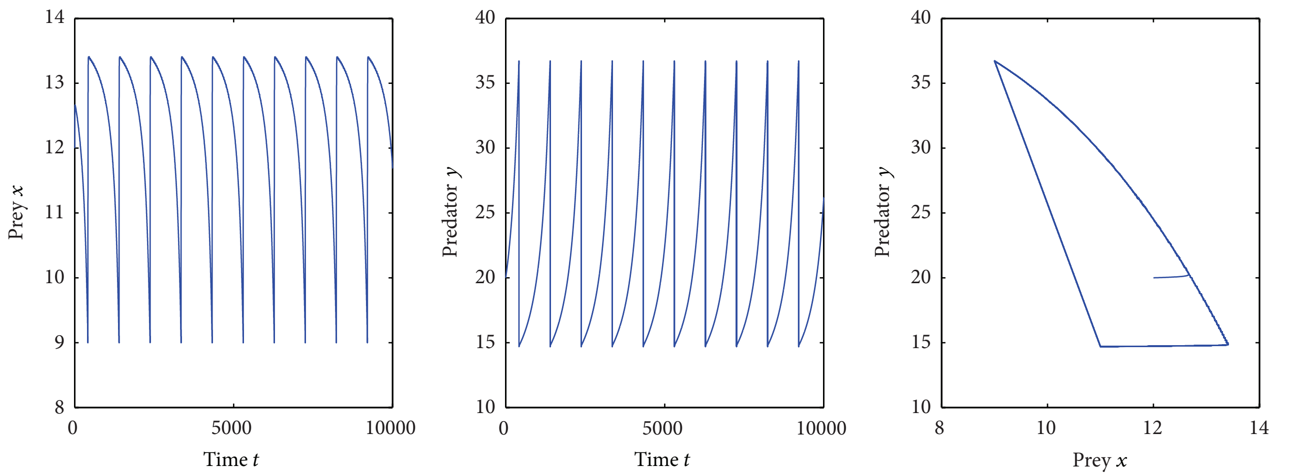

Figure 10: Time series and portrait phase of the system (2) when $r = 0.5$, $K = 15$, $a = 100$, $\mu = 0.8$, $D = 0.038$, $h = 9$, $\tau = 2$, $\beta = 0.6$, and $(x(0), y(0)) = (12, 20)$.

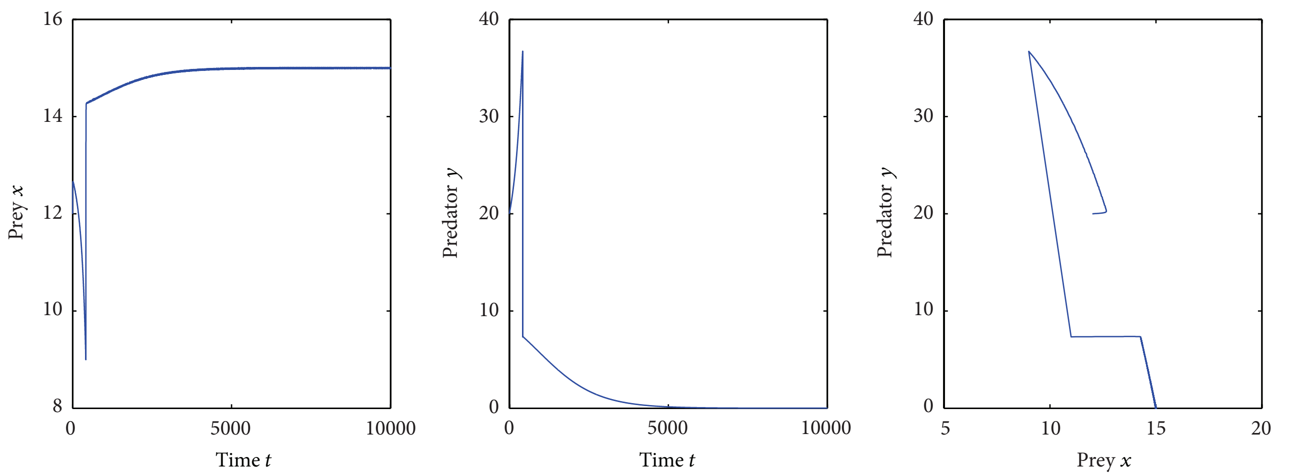

Figure 11: Time series and portrait phase of the system (2) when $r = 0.5$, $K = 15$, $a = 100$, $\mu = 0.8$, $D = 0.038$, $h = 9$, $\tau = 2$, $\beta = 0.8$, and $(x(0), y(0)) = (12, 20)$.

These results illustrate that it demands reasonable control of the harvest yield to form a good ecological environment.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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